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Symmetric Kernel-Based Approach for Elliptic Partial Differential Equation

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Abstract: In this work, two globally supported and positive definite radial kernels, generalized inverse multiquadric and linear Laguerre–Gaussian radial kernels, were used to construct symmetric kernel-based interpolating scheme using Hermite-based symmetric approach for the solution problems involving Hermite's scattered data. Furthermore, two examples on elliptic partial differential equations to illustrate the viability of the symmetric formulation were effectively solved with comparable performance. Results were displayed in form of tables and graphs which present interesting sights for discussions and inference.

Keywords: radial kernel, Hermite's scattered data, symmetric kernel-based interpolation, elliptic PDE, Haar space

1. Introduction

Often time, science and engineering problems are modeled in the form of elliptic partial differential equations. In several physical situations, these problems occur as a result of nature. Fasshauer and McCourt (2016) mentioned some areas such as steady state distribution of heat in the body, harmonic analysis, geometry and more, and they also stated that problems such as acoustic waves can also be managed using the Helmholtz equation. A couple of methods and approaches has been developed by some researchers such as Aziz and Ahmad (2015) to handle these problems using different approaches of the meshless methods based on radial kernels. Fang et al., (2019) used this method in solving integral equations.

The radial kernel method is another alternative numerical approach for higher-dimensional problems of its kind, since it has much many properties which are helpful especially for high order in terms of accuracy and fitting quality (Hastie et al., 2009; Larsson & Fomberg, 2003). The main idea of radial kernels' approach is to approximate the solution as a sum of infinitely many differentiable radial kernels φ whose summand is an appropriate vector times of an appropriate scalar (linear combination). This φ is a function that depends only on the distance from a fix point called the center point x (Fasshauer, 2007). The system matrix in this method denoted $A(\varepsilon)$ is known to be solvable if the matrix $A(\varepsilon)$ is nonsingular (Micchelli, 1984). The radial kernel interpolation methods can also be used for the solution of problems involving elliptic PDE. These methods allow for the interpolation of highly unstructured data. There are few computational setbacks associated with radial kernel methods, which are the structural formulation of the interpolant and the succeeding evaluation of the interpolant function. Given N fixed points to construct an interpolant required inverting the system matrix where the number of rows is $O(N)$ (Wendland, 2002). For globally supported and strictly positive definite radial kernels like the Gaussians, the interpolation matrix is dense according to Yensiri & Skulkhu (2017). According to Fasshauer and McCourt (2016), solving an interpolant for a larger set of centers requires inverting a large, dense, and ill-conditioned system matrix. Thus, the cost of evaluation or computing this interpolant on M data points is of order $O(MN)$.

The aim of this manuscript is to utilize the symmetric kernel-based Hermite interpolation approach proposed by Fasshauer and McCourt (2016) for the numerical solution of elliptic PDEs. The numerical studies' results are obtained by using two different globally supported and strictly positive definite radial kernels, the linear Laguerre–Gaussian (LLG) and the generalized inverse multiquadric (GIMQ) on two computational domains, the uniformly spaced data points and the scattered data points.

2. Preliminaries

In this segment, some definitions and fundamental results relating to the notion of symmetric kernel-based interpolation and elliptic partial differential equation are presented.

2.1. Interpolation problem

Given a set data $\{x_i, f_i\}_{i=1}^{N}$ with $x_i \in \mathbb{R}^d$, $f_i \in \mathbb{R}$, it is required for one to generate a continuous function s in a way that $s(x_i) = f_i, \ \ i = 1, 2, \ldots, N$ is a fundamental mathematics problem (Pazouki, 2012).

A convenient way or method to determine the interpolant s is to look at s as linear combination of certain radial basis kernels $B_i, \ \ i = 1, \ldots, N$. That is,

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$$s(x) = \sum_{i=1}^{N} c_i B_i(x), \qquad x \in \mathbb{R}^d \tag{1}$$

where $B = \{K(\cdot, x_1), \ldots, K(\cdot, x_N)\}$. Solving this problem of interpolation using this assumption as linear combination of the basic functions $B_i(x)$ and the scalars c_i produced a system of linear equations of the form:

$$Bc = f$$

where these entries for the interpolation matrix B are given as $B_{ij} = B_i(x_j),\ i, j = 1, \ldots, N$, $c = (c_1 \ldots c_N)^T$, and $f = (f_1, \ldots, f_N)^T$.

The system matrix assembled from the problem will be solvable if the matrix B is nonsingular. For $d = 1$ implies that one could interpolate an arbitrary data at N distinct set of data points using a polynomial of degree $N - 1$ (Macedo et al., 2009).

Definition 1. Elliptic differential operator: According to Volpert (2011) and Zhao (2016), a linear operator $L : C^2(\Omega) \rightarrow C(\Omega)$ as written in the equation below is called an elliptic differential operator of second order:

$$\mathcal{L}u(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} u(x) + b_0(x) u(x),$$

where the coefficient of the system's matrix $\left[a_{ij}(x)\right] \in R^{d \times d}$ satisfies

$$\exists \alpha > 0, \sum_{i,j=1}^{d} a_{ij}(x) c_i c_j > \alpha ||c||_2^2 \text{ for all } x \in \Omega \textit{ and } c \in R^d.$$

2.2. Interpolation by radial kernels

Radial kernel interpolation is a method in approximation theory for the construction of higher-order accurate interpolants for scattered data up to higher-dimensional spaces. According to Esmaeilbeigi et al. (2018), the interpolation takes the form of a weighted sum of radial kernels. The kernel method is meshless, which means that the data centers must not necessarily lie on a defined grid and does not require the formation of a grid or mesh. It is spectrally accurate for a large numbers of data nodes even in higher dimensions (Fasshauer & McCourt, 2016).

Given a set of data $f = (f(x_1),\ f(x_2),\ \cdots,\ f(x_N))^T \in \mathbb{R}^N$ of function's values obtained from some function say $f : \mathbb{R}^d \rightarrow \mathbb{R}$ at finite set of data points $\Xi = \{x_1, x_2, \cdots, x_N\} \subset \mathbb{R}^d,\quad d \geq 1$, is also given (Marchi & Perracchione, 2018), scattered data interpolation seeks to finds an interpolant function say $s : \mathbb{R}^d \rightarrow \mathbb{R}$ that satisfies

$$s(x_i) = f(x_i), \quad \text{for } i = 1, 2, \cdots, N.$$

The radial kernel interpolation scheme works with kernel functions $\varphi : \mathbb{R}_0^+ \rightarrow \mathbb{R}$, and their interpolant takes the form:

$$s(x) = \sum_{j=0}^{N} c_i \varphi\left(\varepsilon ||x - x_j||\right)$$

where $||\cdot||$ is the Cartesian space norm and ε is the shape parameter. This gives an $N \times N$ linear system:

$$\sum_{j=0}^{N} c_j \varphi\left(\varepsilon ||x_i - x_j||\right) = f(x_i), \quad \text{for } i = 1, 2, \cdots, N$$

this can be rewritten in a vectorial matrix notation as:

$$Ac = f$$

where $A = \varphi\left(\varepsilon ||x_i - x_j||\right)$ is an $N \times N$ matrix and $c = (c_1, c_2, \cdots, c_N)^T$. The matrix A is the interpolation matrix. We note that $\varphi\left(\varepsilon ||x_i - x_j||\right) = \varphi\left(\varepsilon ||x_j - x_i||\right)$ so that $A = A^T$. The interpolant is unique if and only if the matrix A is nonsingular. The existence of the interpolant has been shown in Fasshauer & McCourt (2016).

2.3. Framework

The Haar systems is very fundamental in the formulation of the approximation theory and interpolation. The existence of such system give us the possibility that a unique interpolant (solution) exists from the system. That gives us the first step toward guaranteeing a well-posed problem formulation.

Definition 3: According to Hangelbroek et al. (2014), if a finite-dimensional linear function space $\mathcal{B} \subseteq C(\Omega)$ has a basis $\{B_1, \ldots, B_N\}$, then $\mathcal{B}$ is a Haar space on the domain if

$$\det \mathrm{B} \neq 0$$

for arbitrary set of data $x_1, \ldots, x_N$ in the domain, where B is the system matrix having the entries $(\mathrm{B})_{ij} = \mathrm{B}_i(x_j)$. The set $\{B_1, \ldots, B_N\}$ is refer to as the Haar system.

Theorem 1: The set $\{B_1, \ldots, B_N\}$ of continuous functions on $[a,\ b]$ are said to be a Haar space if and only if any nontrivial linear combination of $B_1, \ldots, B_N$ has at most $N - 1$ zeroes in $(a,\ b)$ (Fasshauer & McCourt, 2016).

Note that if $\Omega \subset \mathbb{R}^d,\ d > 1$, one can no longer guarantee a solution to the system if one chooses the basis different from the data sites. A fact is implied by the Mairhuber–Curtis theorem.

Theorem 2: If $\Omega \subset \mathbb{R}^d, d \geq 2$, contains an interior point, then there exist no Haar spaces of continuous functions except for trivial ones, that is spaces spanned by a single function (Wendland, 2005).

The above theorem harbors two aspects of dimensionality: first of which is the dimension d for the system space on which the data points lie, and second is the dimension N of the space functions $\mathcal{B}$. Consequently, this theorem assures that one cannot manage the basis consisting of more than $N = 1$ functions. There is no surety that the interpolation in the space at N arbitrary points $\mathrm{x_i} \in \mathbb{R}^{\mathrm{d}},\ d \geq 2$, has a unique solution (Hon et al., 2003). Though, if a basis is selected after the data sites are given, and, the conditions given by the theorem are met. This triggers the use of kernel-based methods in higher space dimensions, because they allowed selecting the set $\mathcal{B} = \{K(\cdot, x_1), \ldots, K(\cdot, x_N)\}$ as a basis that is naturally adaptable for the points $\chi = \{\mathrm{x}_1,\ \ldots, \mathrm{x}_\mathrm{N}\}$ so that (1) becomes

$$s(x) = \sum_{i=1}^{N} \mathrm{c_i} K(x, x_\mathrm{i}) = k(x)^\mathrm{T} c, \quad x \in \mathbb{R}^\mathrm{d} \tag{2}$$

and the coefficient $\mathrm{c_i}$ are obtained by solving the linear system;

$$Kc = f$$

After obtaining the coefficients as $c = K^{-1}f$, one can then evaluate the radial kernel interpolant (2) as:

$$s(x) = k(x)^{T}K^{-1}f,$$

where the vector $k(x)^{T} = (K(x, x_1)\ldots K(x, x_N))$. The kernel-based approach can be supported with a good theoretical background. It is an advantage in using positive definite kernels so that the matrix K is positive definite and thus invertible (Hastie et al., 2009).

2.4. Symmetric kernel-based expansion approach

Let us consider the interpolation problem in Section 2.1. And also, at some of the nodes x_k^D, $k = 1, \ldots, N^D$, the values of the derivatives of the interpolating function, essentially given by $(D_k f)(x_k) = Df_k$, where D_k is the differential operator impose on the function at k^{th} node are known (Wendland, 2002). To estimate an approximate value of the function at other locations in the domain apart from the given nodes, the radial Kernels method is presented in the following form:

$$u(x) = \sum_{j=1}^{N} \alpha_j \varphi(||x - \xi||)\big|_{\xi = x_j^I} + \sum_{j=1}^{N^d} \beta_j \left[D_j^{\xi}\varphi(||x - \xi||)\right]_{\xi = x_j^D} \tag{3}$$

where $\varphi(||x - \xi||)$ denotes the kernel function, whose value depends on the distance from an interpolation point x to a fixed point ξ, refers to as the center and D_j^{ξ}, the differential operator acting on the function at x_j^d nodes. In that sense, the differential operator is seen as a function of ξ variable. To determine the interpolation coefficients, the following interpolation conditions were enforced for the function as follows (Krowiak & Podgórski, 2019):

$$\begin{aligned}&\sum_{j=1}^{N} \alpha_j \varphi(||x_i - \xi||)\big|_{\xi = x_j}\\ &+ \sum_{j=1}^{N^d} \beta_j \left[D_j^{\xi}\varphi(||x_i - \xi||)\right]_{\xi = x_j^D} = f_i, \quad i = 1, \ldots, N\end{aligned} \tag{4}$$

as well as for its derivatives:

$$\begin{aligned}&\sum_{j=1}^{N} \alpha_j \left[D_j^{x}\varphi(||x - \xi||)\right]_{\substack{\xi = x_j \\ x = x_i^D}}\\ &+ \sum_{j=1}^{N^d} \beta_j \left[D_j^{x}\left[D_j^{\xi}\varphi(||x - \xi||)\right]_{\xi = x_j^D}\right]_{x = x_i^D} = Df_i \\ &i = 1, \ldots, N^D\end{aligned} \tag{5}$$

In equation (5), D_j^x represents the differential operator same as D_j^{ξ}, but now operating on the kernel is considered as a function of the data x variable. This causes the system matrix of coefficient in equations (4) and (5) to be a symmetric matrix, which can be put together as described in the following matrix notation:

$$\begin{bmatrix} A & A_{D^{\xi}} \\ A_{D^x} & A_{D^x D^{\xi}} \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} f \\ Df \end{bmatrix} \tag{6}$$

where:

$$A_{i,j} = \varphi(||x_i - \xi||)|_{\xi = x_j}, \quad i, j = 1, \ldots, N$$

$$(A_{D^{\xi}})_{i,j} = [D^{\xi}\varphi(||x_i - \xi||)]_{\xi = x_j^D}, \quad i = 1, \ldots, N, \quad j = 1, \ldots, N^D$$

$$(A_{D^{x}})_{i,j} = [D^{x}\varphi(||x - \xi||)]_{\substack{\xi = x_j \\ x = x_i^D}}, \quad i = 1, \ldots, N^D, \quad j = 1, \ldots, N$$

$$(A_{D^x D^{\xi}})_{i,j} = \left[D^x [D^{\xi}\varphi(||x - \xi||)]_{\xi = x_j^D}\right]_{x = x_i^D}, \quad i, j = 1, \ldots, N^D$$

α, β are vectors representing the interpolation coefficients, while f, Df are function values and their corresponding derivative values. In order to obtain the interpolation coefficients for the system, the system in equation (6) has to be solved yielding:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} A & A_{D^{\xi}} \\ A_{D^x} & A_{D^x D^{\xi}} \end{bmatrix}^{-1} \cdot \begin{bmatrix} f \\ Df \end{bmatrix}$$

The problem of existence of the solution of the system depends on the type of kernel used (Krowiak & Podgórski, 2019; Speckbacher & Balazs, 2019).

2.5. Shape parameter

In this work, the brute force method was used to compute a suitable estimate for the shape parameter ε. The brute force method consists of performing various interpolation experiments using different values of the shape parameter ε (Fasshauer, 2007). The best value of the shape parameter is the one that best minimizes the interpolation error. This is achieved by plotting the interpolation error against the shape parameter. The minimum point on the curve gives the optimal value of the shape parameter ε. The graphs of both RMS-error and Max-error against different values of the shape parameter shall be plotted, and the value at the minimum point is an estimate use for the experiment (Galichi et al., 2022).

3. Numerical Studies

Problem 1. Consider the following elliptic PDE:

$$\nabla^2 u + xu_x + yu_y = (4x^2 + y^2) = f(x, y)$$

where $f(x, y)$ and the boundary conditions are obtained from the same solution of the following equation $u(x, y) = \exp(-(x^2 + 0.5y^2))$.

The numerical results displayed in terms of RMS-error, Max-error, and the reciprocal of the condition number. The implementation is done with $N = 1089$ on three types of data point locations using linear Matern and linear Laguerre–Gaussians.

Problem 2. Consider the following elliptic PDE:

$$\nabla^2 u + Ru_x = f(x, y)$$

where $f(x, y)$ and the Dirichlet boundary condition are computed from the same solution as shown in $u(x, y) = \exp(-0.5Rx)\sin(0.5Ry)$

The numerical results are displayed in terms of RMS-error, Max-error, and the reciprocal of the condition number. The implementation is done with $N = 1089$ on three types of data point locations using linear Matern and linear Laguerre–Gaussians.

4. Discussion

The method was implemented using $N = 1089$ nodal points. The RMS-error and Max-error norms were observed using same value of N with ε ranging from 2 to 6 for problem 1 as shown in Table 1. The computations were carried out using linear Matern and linear Laguerre–Gaussian kernels on a domain containing Chebyshev's and uniformly types of data set each. For problem 2, the implementation was executed using the two kernels on scattered data site. The optimal value of shape parameter was found to be 6 for linear Matern and 4 for linear Laguerre Gaussian for problem 1 on each of the data type. And the optimal shape parameter was found to be 1 for linear Matern and 5 for linear Laguerre–Gaussian for problem 2 on uniformly spaced data type, as shown in Table 2. It was seen that a good approximate solution is obtained by applying this method as the root mean square error is as minimal as $o(10^{-10})$ using the Linear Matern on Chebyshev's data points at shape parameter value of 6 in problem 1 and in problem 2, and the best approximate solution is obtained using linear Laguerre–Gaussian at the shape parameter value of 5 as shown in Tables 1 and 2. It was interesting to note in the course of the experiments that this method has advantage over the other conventional methods, this is because, at the ill-conditioned state of it system matrix as seen in Tables 1 and 2, the method still performed favorable on both regular and irregular domains. Figures 1 and 2 showed that the method perform well with minimum region of

Table 1
Error norms and reciprocal of condition for problem

Mesh	RBF	ε	RMS_Error	Max_Error	RCOND
Solution on a domain containing Chebyshev's type of data points					
1089	Linear Matern	2	$\times 10^{-7}$ 5.124650	$\times 10^{-6}$ 1.959598	$\times 10^{-22}$ 5.476490
1089	Linear Matern	3	$\times 10^{-8}$ 1.844909	$\times 10^{-8}$ 6.908423	$\times 10^{-21}$ 1.819013
1089	Linear Matern	4	$\times 10^{-8}$ 3.370381	$\times 10^{-8}$ 9.947352	$\times 10^{-22}$ 5.679957
1089	Linear Matern	5	$\times 10^{-7}$ 2.465424	$\times 10^{-7}$ 7.233035	$\times 10^{-23}$ 9.198231
1089	Linear Matern	6	$\times 10^{-10}$ 2.683164	$\times 10^{-9}$ 3.194507	$\times 10^{-19}$ 1.766803
Solution on a domain containing Chebyshev's type of data points					
1089	LL Gaussians	2	$\times 10^{-6}$ 2.551212	$\times 10^{-5}$ 1.621135	$\times 10^{-22}$ 2.225880
1089	LL Gaussians	3	$\times 10^{-8}$ 3.059825	$\times 10^{-7}$ 3.324615	$\times 10^{-22}$ 9.400252
1089	LL Gaussians	4	$\times 10^{-8}$ 8.815699	$\times 10^{-6}$ 2.291940	$\times 10^{-22}$ 7.015095
1089	LL Gaussians	5	$\times 10^{-7}$ 5.962984	$\times 10^{-5}$ 2.030384	$\times 10^{-22}$ 3.175404
1089	LL Gaussians	6	$\times 10^{-6}$ 1.598992	$\times 10^{-5}$ 6.308704	$\times 10^{-21}$ 1.535246
Solution on a domain containing uniformly spaced data points					
1089	Linear Matern	2	$\times 10^{-5}$ 1.530104	$\times 10^{-5}$ 5.809703	$\times 10^{-23}$ 4.202345
1089	Linear Matern	3	$\times 10^{-8}$ 2.271218	$\times 10^{-8}$ 9.542311	$\times 10^{-21}$ 1.069985
1089	Linear Matern	4	$\times 10^{-8}$ 1.531907	$\times 10^{-8}$ 5.061285	$\times 10^{-21}$ 1.443477
1089	Linear Matern	5	$\times 10^{-9}$ 7.692016	$\times 10^{-8}$ 1.783898	$\times 10^{-22}$ 4.377331
1089	Linear Matern	6	$\times 10^{-9}$ 6.858826	$\times 10^{-8}$ 2.558163	$\times 10^{-22}$ 5.264407
Solution on a domain containing uniformly spaced data points					
1089	LL Gaussians	2	$\times 10^{-6}$ 2.551212	$\times 10^{-5}$ 1.621135	$\times 10^{-22}$ 2.225880
1089	LL Gaussians	3	$\times 10^{-8}$ 3.059825	$\times 10^{-7}$ 3.324615	$\times 10^{-22}$ 9.400252
1089	LL Gaussians	4	$\times 10^{-8}$ 8.815699	$\times 10^{-6}$ 2.291940	$\times 10^{-22}$ 7.015095
1089	LL Gaussians	5	$\times 10^{-7}$ 5.962984	$\times 10^{-5}$ 2.030384	$\times 10^{-22}$ 3.175404
1089	LL Gaussians	6	$\times 10^{-6}$ 1.598992	$\times 10^{-5}$ 6.308704	$\times 10^{-21}$ 1.535246

Table 2
Error norms and reciprocal of condition for problem

N	ε	RBF	RMS_Error	Max_Error
Numerical solution using linear Matern on scattered data points				
1089	1	Linear Matern	$\times 10^{-7}$ 6.936248	$\times 10^{-6}$ 4.996954
1089	0.75	Linear Matern	$\times 10^{-6}$ 1.993316	$\times 10^{-6}$ 7.197521
1089	0.5	Linear Matern	$\times 10^{-6}$ 4.438264	$\times 10^{-5}$ 1.681790
1089	0.25	Linear Matern	$\times 10^{-6}$ 2.287123	$\times 10^{-6}$ 8.336800
1089	0.1	Linear Matern	$\times 10^{-6}$ 2.287123	$\times 10^{-6}$ 8.336800
Numerical solution using linear Laguerre–Gaussians on scattered data points				
1089	25	Laguerre–Gaussians	$\times 10^{-7}$ 1.179371	$\times 10^{-7}$ 7.840436
1089	20	Laguerre–Gaussians	$\times 10^{-8}$ 8.314964	$\times 10^{-7}$ 5.449712
1089	15	Laguerre–Gaussians	$\times 10^{-8}$ 3.250818	$\times 10^{-7}$ 2.122126
1089	10	Laguerre–Gaussians	$\times 10^{-9}$ 4.132460	$\times 10^{-8}$ 3.576482
1089	5	Laguerre–Gaussians	$\times 10^{-9}$ 4.132460	$\times 10^{-8}$ 3.576482

Figure 1
(a) Numerical approximation and (b) error norm for problem 1 for $N = 1089$

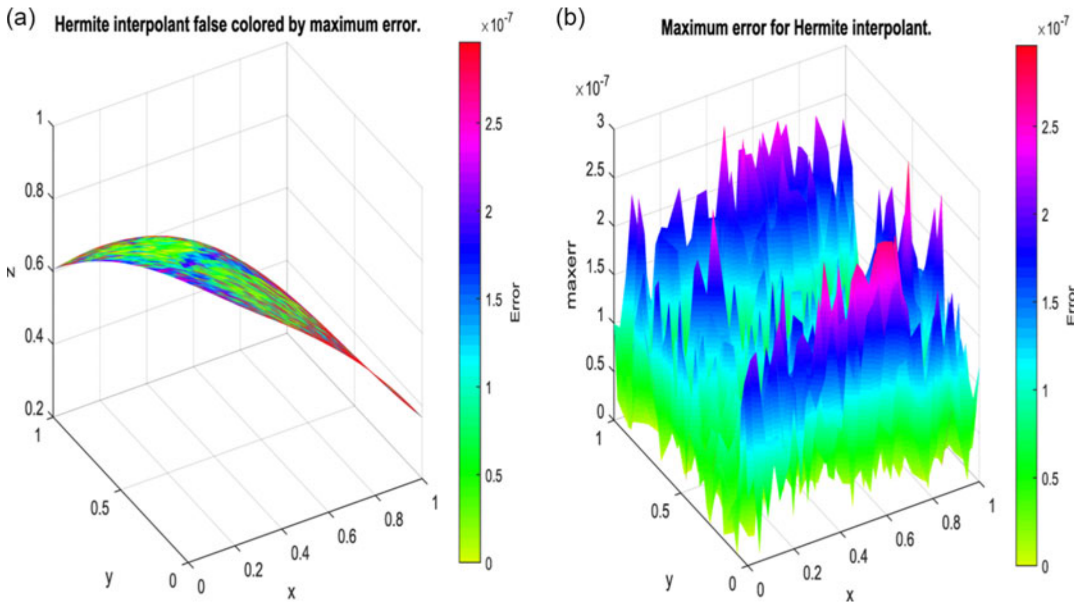
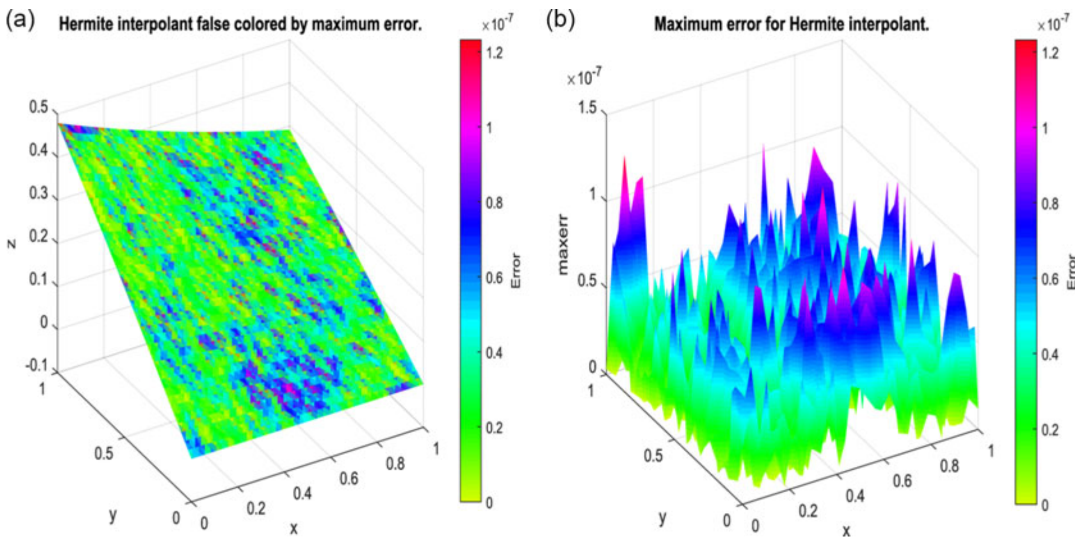


Figure 2
(a) Numerical approximation and (b) error norm for problem 2 for $N = 289$



errors. Figures 1(a) and 2(a) are the numerical approximations for problem 1 and 2, while Figures 1(b) and 2(b) are the error norms showing regions of errors with false colors for problem 1 and 2, respectively. It was also noticed that all the three set of data types used the uniformly spaced data points, the scattered data points and the Chebyshev's type of data point gave a fair approximation but the Chebyshev's data-type boasts of superiority for both the linear Matern and the linear Laguerre–Gaussian.

5. Conclusion

In this article, the symmetric kernel-based interpolation approach for functions values located at unstructured nodes is illustrated. Consequently, the approach has been shown to be capable of solving problems modeled in the form of elliptic partial differential equation. Attention was given to the appropriate value of the scaled parameter called the shape parameter included in the kernels. One can see that the best approximate solution can be obtained by testing different values of the shape parameter and also finding the suitable combination of the kernel to use and the type of data site. It was concluded that Hermite interpolation with Matern kernel and the linear Laguerre–Gaussians kernel using symmetric formulation present a strong alternative for modeling solutions to problems involving elliptic partial differential equation.

The fact that elliptic partial differential equation arises in nature in several real-life situations, such as found in the behavior of sound, heat, electrostatics, fluid flow, and elasticity, the symmetric kernel-based interpolation method will help create a model that will give full description of their behavior using an available data.

Recommendations

The results obtained in this work show that the Hermite Scattered Data Interpolation Method using Matern and linear Laguerre–Gaussian kernels can be used in solving elliptic partial differential equations. It is also recommended that the optimal estimate for the shape parameter should be obtained when using kernels that contain such parameters.

The following are recommended for further research work:

i. Other radial kernels and more suitable error indicators can be used for formulation of Hermite symmetric interpolation approach.
ii. The article was able to use only a rectangular computational domain with three sets of data points. This can be extended to other computational domains especially the irregular domains in order to check behavior of the solution.
iii. This research applied Hermite interpolation to two-dimensional problems, but this method can be modified for higher-dimensional problems and time-dependent problems.

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Conflicts of Interest

The author declares that he has no conflicts of interest to this work.

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