Received: 12 September 2025 | Revised: 23 October 2025 | Accepted: 29 October 2025 | Published online: 12 March 2026

RESEARCH ARTICLE

Journal of Data Science and Intelligent Systems
2026, Vol. XX(XX) 1–17
DOI: 10.47852/bonviewJDSIS62027651

BON VIEW PUBLISHING

HyperCloudy Number and (m,n)-SuperHyperCloudy Number

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Abstract: Uncertainty modeling is fundamental to decision-making across diverse domains, and numerous frameworks, such as Fuzzy Sets, Rough Sets, Hesitant Fuzzy Sets, and Plithogenic Sets, have been developed to capture different facets of imprecision. Among these, HyperFuzzy Sets and their recursive generalization, SuperHyperFuzzy Sets, assign set-valued membership degrees at multiple hierarchical levels to represent uncertainty more richly. A *Cloudy Number* assigns to each real value a closed interval within $[0,1]$, modeling uncertainty via lower and upper membership levels. Cloudy Numbers have deep connections with Fuzzy Numbers and real numbers, and substantial related research has been conducted. However, the counterparts of HyperFuzzy Numbers and SuperHyperFuzzy Numbers in the setting of Cloudy Numbers remain unexplored. In this paper, we introduce two new constructs: the *HyperCloudy Number* and the (m,n)-*SuperHyperCloudy Number*. Inspired by the HyperFuzzy Number and the (m,n)-SuperHyperFuzzy Number, these concepts hierarchically generalize the existing notion of a Cloudy Number. We establish their fundamental properties and demonstrate their applicability using concrete examples.

Keywords: Fuzzy Set, HyperFuzzy Set, SuperHyperFuzzy Set, Cloudy Number, cloud

1. Introduction

1.1. Fuzzy, HyperFuzzy, and SuperHyperFuzz Sets

Uncertainty modeling is essential in decision-making across many fields. A *Fuzzy Set* associates each element x of a universe U with a membership degree $\mu_A(x) \in [0,1]$, allowing partial inclusion rather than a strict binary choice [1–3]. Such set has been applied in areas ranging from control theory and artificial intelligence to graph algorithms and topological analysis. Numerous variants have been proposed, including Intuitionistic Fuzzy Sets [4, 5], Bipolar Fuzzy Sets [6, 7], Vague Sets [8], Neutrosophic Sets [9–12], Quadripartitioned Neutrosophic Sets [13, 14], and Pentapartitioned Neutrosophic Sets [15–17]. Moreover, owing to their ease of real-world applicability, Fuzzy Sets and their extensions have been studied across many fields, including decision-making [6, 18, 19], control theory [20–22], and machine learning [23–25], and are used in practice. Going further, a *HyperFuzzy Set* assigns to each element $x \in U$ a nonempty subset of $[0,1]$, thereby capturing uncertainty about the membership degree itself [26–30]. The (m,n)-*SuperHyperFuzzy Set* extends this idea by mapping each nonempty m-tuple of universe elements to a family of nonempty n-level membership subsets, modeling multilayered ambiguity [31]. SuperHyperFuzzy Sets model hierarchical, multilevel uncertainty with set-valued memberships, aggregating heterogeneous evidence, preserving conflict, enabling explainable nesting, and generalizing Fuzzy and HyperFuzzy frameworks for decisions.

Table 1 presents an overview of Fuzzy, HyperFuzzy, and (m,n)-SuperHyperFuzzy sets.

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1.2. Fuzzy Number and Cloudy Number

Fuzzy Sets, HyperFuzzy Sets, and SuperHyperFuzzy Sets have related notions known as Fuzzy Numbers, HyperFuzzy Numbers, and SuperHyperFuzzy Numbers. These concepts extend the idea of fuzziness to numerical representations, allowing uncertain or imprecise values to be treated as numbers rather than mere set memberships. Because research on Fuzzy Numbers, HyperFuzzy Numbers, and SuperHyperFuzzy Numbers has been proven to be important, several related numerical frameworks have also been proposed, including Neutrosophic Numbers [32–34], Plithogenic Numbers [35, 36], Intuitionistic Fuzzy Numbers [19, 37], Z-Numbers [38, 39], and D-Numbers [40, 41]. These concepts have been applied widely across many domains, including decision-making and linear programming, with a range of practical uses [42–46].

A *Cloudy Number* represents each real value $r \in \mathbb{R}$ by a closed interval $[\underline{\mu}(r), \overline{\mu}(r)] \subseteq [0,1]$, reflecting lower and upper bounds on membership uncertainty [47–49]. Cloudy Numbers capture lower–upper membership bounds, enabling robust decisions under imprecision; they preserve uncertainty structure, support interval arithmetic, aggregate heterogeneous evidence, and yield conservative inferences for risk-sensitive optimization and analysis.

The differences between Fuzzy Numbers and Cloudy Numbers are summarized in Table 2.

1.3. Our contributions

From the above discussion, it is clear that research on Fuzzy Numbers, HyperFuzzy Numbers, SuperHyperFuzzy Numbers, and Cloudy Numbers is of significant importance. Although Cloudy Numbers are closely related to Fuzzy Numbers and have been studied in various contexts, analogous extensions corresponding to HyperFuzzy Numbers and SuperHyperFuzzy Numbers have

Table 1
Concise overview of Fuzzy, HyperFuzzy, and *(m,n)*-SuperHyperFuzzy Sets

	Fuzzy Set	HyperFuzzy Set	(m,n)-SuperHyperFuzzy Set
Universe/domain	U	U	$\mathscr{P}_m^*(U)$ (nonempty m-level subsets)
Membership mapping	$\mu : U \to [0,1]$	$\widetilde{\mu} : U \to \mathscr{P}([0,1]) \setminus \{\emptyset\}$	$\widetilde{\mu}_{m,n} : \mathscr{P}_m^*(U) \to \mathscr{P}(\mathscr{P}_n([0,1])) \setminus \{\emptyset\}$
Value per input	Single grade $\mu(x)$	Nonempty set of grades $\widetilde{\mu}(x) \subseteq [0,1]$	Nonempty family of n-level g rade-sets for each X
Uncertainty captured	Partial membership (point-valued)	Membership ambiguity (set-valued)	Hierarchical, multi-layer ambiguity over inputs and grades
α-level notion	$A_\alpha = \{x \in U \mid \mu(x) \geq \alpha\}$	$\{x \in U \mid \exists u \in \widetilde{\mu}(x) : u \geq \alpha\}$	$\{X \in \mathscr{P}_m^*(U) \mid \exists V \in \widetilde{\mu}_{m,n}(X),\ \exists u \in \bigcup V :\ u \geq \alpha\}$
Relations/special cases	—	Reduces to Fuzzy if each $\widetilde{\mu}(x)$ is a singleton	Reduces to HyperFuzzy m=0, n=1; to Fuzzy if values further singletonize
Typical use	Graded predicates; smooth gradations	Aggregating conflicting sources without collapse	Modeling nested evidence/multiagent, multicriteria uncertainty

not yet been explored. To fill this gap, in this work, we introduce two novel constructs: the *HyperCloudy Number* and the (m,n)-*SuperHyperCloudy Number*. Inspired by their fuzzy counterparts, these new definitions embed iterative and hierarchical uncertainty levels into the Cloudy Number framework. We establish their foundational properties, discuss their algebraic behavior, and demonstrate their applicability through illustrative and concrete examples.

Table 3 presents a concise overview of Cloudy, HyperCloudy, and (m,n)-SuperHyperCloudy Numbers. HyperCloudy and (m,n)-SuperHyperCloudy models capture hierarchical uncertainty via interval families, preserve conflict, support conservative inference, enable aggregation, and unify fuzzy and hyperfuzzy reasoning for decision-making.

1.4. Structure of this paper

This section outlines the organization of this paper. Section 2 defines well-established concepts such as Cloudy Numbers and Fuzzy Sets. Section 3 introduces new notions called HyperCloudy Numbers and SuperHyperCloudy Numbers, and examines their properties and characteristics. Section 4 presents the conclusion of this paper.

2. Preliminaries

We introduce here the fundamental definitions and notation used throughout this paper. Unless stated otherwise, all sets under consideration are finite.

Table 2
Side-by-side comparison of Fuzzy Numbers and Cloudy Numbers

	Fuzzy Number	Cloudy Number
Domain/object	Real line $\mathbb{R}$; imprecise real modeled by a Fuzzy Set.	Real line $\mathbb{R}$; imprecise real modeled by interval-valued membership levels.
Mapping (input $\to$ output)	$\mu_A : \mathbb{R} \to [0,1], \quad x \mapsto \mu_A(x).$	$x : \mathbb{R} \to \{[\ell,u] \subseteq [0,1]\}, \quad t \mapsto x(t) = [\underline{x}(t), \overline{x}(t)].$
Core regularity	Normality: $\max_x \mu_A(x) = 1$; Convexity: $\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$; Upper semicontinuity; compact support.	*Coverage*: $$(0,1) \subseteq \bigcup_{t\in\mathbb{R}} [\underline{x}(t), \overline{x}(t)] \subseteq [0,1];$$ *Normality*: $$\sup_t \overline{x}(t) = 1,\ \inf_t \underline{x}(t) = 0.$$
Uncertainty representation	Single grade per input x (point value in $[0,1]$).	Interval of plausible grades per input t (lower–upper bounds).
α-level sets	$$A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$$ (nested closed intervals).	$$T_\alpha = \{t \in \mathbb{R} \mid \underline{x}(t) \leq \alpha \leq \overline{x}(t)\}$$ (times/inputs where α is admissible).
Typical operations	Extension principle via α-cuts; t-norm/t-conorm-based aggregation; fuzzy arithmetic by interval propagation over A_α.	Interval arithmetic on $[\underline{x}, \overline{x}]$; conservative/robust aggregation preserving lower–upper uncertainty.
Interpretation	Suited for smooth gradation around a modal value; precise membership at each x.	Suited for epistemic variability or source disagreement; explicitly records imprecision range.
Relationship	Special case of a Cloudy Number by setting $\underline{x}(t) = \overline{x}(t) = \mu_A(t)$.	Can induce families of Fuzzy Numbers by choosing a selector $\mu(t) \in [\underline{x}(t), \overline{x}(t)]$ (e.g., optimistic/pessimistic/median).
Common use cases	Control, optimization, signal processing with smooth membership profiles.	Risk-sensitive decision-making, fusion of heterogeneous evidence, robust analysis under bounded uncertainty.

Table 3
Concise summary of Cloudy, HyperCloudy, and *(m,n)*-SuperHyperCloudy Numbers

Object	Mapping (input $\to$ output)	Coverage/normality conditions	Notes/relations
Cloudy Number	$x:\mathbb{R}\to\{[\ell,u]\subseteq[0,1]\},\quad t\mapsto x(t)=[\underline{x}(t),\overline{x}(t)]$.	$(0,1)\subseteq\bigcup_{t\in\mathbb{R}}x(t)\subseteq[0,1]$; $\sup_t\overline{x}(t)=1$, $\inf_t\underline{x}(t)=0$.	Lower/upper membership bounds for each real t; a single interval per input.
HyperCloudy Number	$\mathfrak{X}:\mathbb{R}\to\{\mathscr{I}\neq\emptyset\},\quad \mathfrak{X}(t)=\{[\underline{x}_i(t),\overline{x}_i(t)]\}_{i\in I_t}$.	$(0,1)\subseteq\bigcup_{t,i}[\underline{x}_i(t),\overline{x}_i(t)]\subseteq[0,1]$; $\sup_{t,i}\overline{x}_i(t)=1$, $\inf_{t,i}\underline{x}_i(t)=0$.	Multisource interval family at each t. If $\lvert I_t\rvert=1\Rightarrow$ Cloudy; if each interval collapses ($\underline{x}_i=\overline{x}_i$) $\Rightarrow$ Hyperfuzzy.
(m,n)-SuperHyperCloudy Number	$\mathfrak{X}^{(m,n)}:\mathscr{P}^*_m(\mathbb{R})\to\{\mathscr{I}\neq\emptyset\}$, $\mathfrak{X}^{(m,n)}(X)=\{[\underline{x}_{i,j}(X),\overline{x}_{i,j}(X)]\}_{i\in I_X,\ j\in J_{X,i}}$.	$(0,1)\subseteq\bigcup_{X,i,j}[\underline{x}_{i,j}(X),\overline{x}_{i,j}(X)]\subseteq[0,1]$; $\sup_{X,i,j}\overline{x}_{i,j}(X)=1$, $\inf_{X,i,j}\underline{x}_{i,j}(X)=0$.	Hierarchical uncertainty: input via $\mathscr{P}^*_m(\mathbb{R})$, output via n-level interval families. Singleton embedding/flattening $\Rightarrow$ HyperCloudy; point-degenerate endpoints $\Rightarrow$ (m,n)-SuperHyperFuzzy.

2.1. Fuzzy, HyperFuzzy, and SuperHyperFuzzy Sets-theoretic foundations

A Fuzzy Set associates each element x of a universe U with a membership degree $\mu_A(x)\in[0,1]$, allowing partial inclusion rather than a strict binary choice [1–3]. These concepts are extended through the notions of the *Power Set* and the *n-th Power Set*, leading to the definitions of the HyperFuzzy Set and the SuperHyperFuzzy Set. The definitions of the Power Set, n-th Power Set, Fuzzy Set, HyperFuzzy Set, and SuperHyperFuzzy Set are provided below.

Definition 1 (Universe). Let U be a nonempty finite set, called the *universe* or *base set*. Every subsequent construction—powersets, hyperstructures, and so on—is built upon U.

Definition 2 (Power Set) (cf. [50]). The Power Set of U is

$$\mathscr{P}(U) = \{A \mid A\subseteq U\}.$$

Definition 3 (Iterated powerset) [51]. For each integer $n\geq 1$, the n-fold iterated powerset of U is defined by

$$\mathscr{P}^1(U)=\mathscr{P}(U),\quad \mathscr{P}^{n+1}(U)=\mathscr{P}\big(\mathscr{P}^n(U)\big).$$

If one wishes to exclude the empty set at each iteration, replace $\mathscr{P}$ with

$$\mathscr{P}^*(X) = \mathscr{P}(X)\setminus\{\emptyset\}.$$

Example 1 (Iterated powerset: menu planning). Let $U=\{\text{tomato},\text{cheese},\text{dough},\text{lettuce},\text{olive oil}\}$. Then, $\mathscr{P}^1(U)$ lists all *dishes* (e.g., $\{\text{tomato},\text{cheese},\text{dough}\}=$ pizza, $\{\text{lettuce},\text{olive oil}\}=$ salad). $\mathscr{P}^2(U)$ lists *menus*, e.g., $M=\{\{\text{tomato},\text{cheese},\text{dough}\},\{\text{lettuce},\text{olive oil}\}\}$. $\mathscr{P}^3(U)$ lists *meal plans*, e.g.,

$$P=\{M,\{\{\text{cheese},\text{dough}\},\{\text{tomato},\text{olive oil}\}\}\}.$$

Thus, $\mathscr{P}^1(U)\to\mathscr{P}^2(U)\to\mathscr{P}^3(U)$ models dishes $\to$ menus $\to$ plans.

Example 2 (Iterated powerset: burgers). Let $U=\{\text{bun},\text{patty},\text{lettuce},\text{tomato},\text{cheese},\text{pickles}\}$. $\mathscr{P}^1(U)$: single *burgers*, e.g., $\{\text{bun},\text{patty},\text{cheese}\}$ (cheeseburger), $\{\text{bun},\text{patty},\text{lettuce},\text{tomato}\}$ (classic). $\mathscr{P}^2(U)$: *combos*, e.g.,

$$(C=\{\{\text{bun},\text{patty},\text{cheese}\},\{\text{bun},\text{patty},\text{lettuce},\text{tomato}\}\}.$$

$\mathscr{P}^3(U)$: *family plans*, e.g.,

$$(F=\{C,\{\{\text{bun},\text{patty},\text{pickles}\},\{\text{bun},\text{patty},\text{cheese},\text{pickles}\}\}\}.$$

Hence, $\mathscr{P}^1(U)\to\mathscr{P}^2(U)\to\mathscr{P}^3(U)$ captures burgers $\to$ combos $\to$ plans.

Definition 4 (Fuzzy Set) [1, 52]. A Fuzzy Set F on a universe U is specified by a membership function

$$\mu_F: U\longrightarrow[0,1],$$

so that each element $x\in U$ is assigned a degree of membership $\mu_F(x)$.

Example 3 (Fuzzy Set). Let the universe be $U=\{a,b,c\}$. Define the membership function

$$\mu_F(a)=0.2,\qquad \mu_F(b)=0.7,\qquad \mu_F(c)=1.0.$$

Then, $F=\{(a,0.2),(b,0.7),(c,1.0)\}$ is a Fuzzy Set on U.

Definition 5 (Fuzzy relation) [53, 54]. Let F be a Fuzzy Set on U. A *fuzzy relation* R on U is a map

$$R\colon U\times U\longrightarrow[0,1],$$

satisfying

$$R(x,y)\ \leq\ \min\{\mu_F(x),\mu_F(y)\}\quad \text{for all } x,y\in U.$$

Example 4 (Fuzzy relation). On $U\times U$ (with row/column order a,b,c), define

$$R=\begin{pmatrix}0.20.20.2\\0.20.70.6\\0.20.61.0\end{pmatrix}.$$

Because, e.g., $R(b,c)=0.6\leq\min\{\mu_F(b),\mu_F(c)\}=\min\{0.7,1.0\}=0.7$ and similarly for all pairs, the constraint $R(x,y)\leq\min\{\mu_F(x),\mu_F(y)\}$ holds.

Definition 6 (HyperFuzzy Set) [26, 29]. A *HyperFuzzy Set* $\widetilde{F}$ on U is given by a function

$$\widetilde{\mu}: U\longrightarrow\mathscr{P}([0,1])\setminus\{\emptyset\},$$

where for each $x\in U$, the nonempty subset $\widetilde{\mu}(x)\subseteq[0,1]$ represents all possible membership grades of x.

Example 5 (Restaurant quality as a HyperFuzzy Set). Let

$$U=\{\text{RestoA},\text{RestoB},\text{RestoC}\}$$

be a collection of three restaurants. Three independent critics—evaluating *taste*, *service*, and *ambiance*—assign a fuzzy score in $[0,1]$ to indicate the degree to which a restaurant is "high-quality." Instead of collapsing these into a single value, we form a HyperFuzzy Set (cf. Definition 6) by collecting all reported scores:

$$\widetilde{\mu}: U\ \longrightarrow\ \mathscr{P}([0,1])\setminus\{\emptyset\},$$

with

$$\widetilde{\mu}(\text{RestoA}) = \{0.85_{(\text{taste})}, 0.75_{(\text{service})}, 0.90_{(\text{ambiance})}\},$$

$$\widetilde{\mu}(\text{RestoB}) = \{0.70_{(\text{taste})}, 0.65_{(\text{service})}, 0.80_{(\text{ambiance})}\},$$

$$\widetilde{\mu}(\text{RestoC}) = \{0.60_{(\text{taste})}, 0.55_{(\text{service})}, 0.70_{(\text{ambiance})}\}.$$

Here, each set $\widetilde{\mu}(x) \subseteq [0,1]$ gathers all plausible membership grades for restaurant x, reflecting the different perspectives of the critics. This hyperfuzzy representation preserves the full range of expert evaluations rather than forcing a single aggregated score.

Definition 7 ((m,n)-SuperHyperFuzzy Set) [55]. Fix integers $m,n \geq 0$. Define

$$\mathscr{P}_m^*(U) = (\underbrace{\mathscr{P}^* \circ \cdots \circ \mathscr{P}^*}_{m \text{ times}})(U),$$

$$\mathscr{P}_n^*([0,1]) = (\underbrace{\mathscr{P}^* \circ \cdots \circ \mathscr{P}^*}_{n \text{ times}})([0,1]),$$

where $\mathscr{P}^*(X) = \mathscr{P}(X) \setminus \{\emptyset\}$. An (m,n)-*SuperHyperFuzzy Set* on U is a mapping

$$\widetilde{\mu}_{m,n} : \mathscr{P}_m^*(U) \longrightarrow \mathscr{P}\big(\mathscr{P}_n([0,1])\big) \setminus \{\emptyset\},$$

which assigns each nonempty m-level subset of U a nonempty family of n-level membership-value sets, thereby capturing hierarchical uncertainty.

Example 6 (Consumer satisfaction as a $(2,2)$-SuperHyperFuzzy Set) (cf. [56]). Let

$$U = \{\text{Battery, Camera, Screen}\}$$

be a set of smartphone features. We take $m = n = 2$, so our domain is $\mathscr{P}_2^*(U)$, the nonempty subsets of U obtained by two iterations of the nonempty-powerset operator. Concretely, we focus on the three two-feature subsets:

$$\{\text{Battery, Camera}\}, \quad \{\text{Battery, Screen}\}, \quad \{\text{Camera, Screen}\}.$$

For each such pair X, three user groups—novice, casual, and expert—provide interval-valued satisfaction scores (in $[0,1]$) reflecting their uncertainty. We assemble these into a $(2,2)$-SuperHyperFuzzy Set

$$\widetilde{\mu}_{2,2} : \mathscr{P}_2^*(U) \longrightarrow \mathscr{P}\big(\mathscr{P}_2([0,1])\big) \setminus \{\emptyset\},$$

by defining for each X:

$$\widetilde{\mu}_{2,2}(X) = \big\{\{\ell_{g,1}(X), \ell_{g,2}(X)\} \;\big|\; g \in \{\text{Novice, Casual, Expert}\}\big\},$$

where the two-element sets $\{\ell_{g,1}, \ell_{g,2}\} \subseteq [0,1]$ are the hyperfuzzy scores from group g. For example:

$$\widetilde{\mu}_{2,2}(\{\text{Battery, Camera}\}) = \{\{0.70, 0.80\}, \{0.65, 0.75\}, \{0.80, 0.90\}\},$$
$$\widetilde{\mu}_{2,2}(\{\text{Battery, Screen}\}) = \{\{0.60, 0.70\}, \{0.55, 0.65\}, \{0.75, 0.85\}\},$$
$$\widetilde{\mu}_{2,2}(\{\text{Camera, Screen}\}) = \{\{0.65, 0.85\}, \{0.60, 0.80\}, \{0.80, 0.95\}\}.$$

Each inner set $\{a,b\}$ is a nonempty subset of $[0,1]$, modeling the group's interval of plausible satisfaction. The outer braces collect these hyperfuzzy values into a nonempty family for each two-feature subset, thus capturing hierarchical uncertainty over both features and user groups.

2.2. Fuzzy-based numerical constructs

We now introduce three increasingly general notions of "numbers" that incorporate various levels of graded uncertainty. A (finite) Fuzzy Number is a Fuzzy Set on $\mathbb{R}$ whose membership function is normal, convex, and upper semicontinuous and has compact support, modeling imprecise real values [57, 58].

Definition 8 (Fuzzy Number) [59]. A Fuzzy Number is a Fuzzy Set A on $\mathbb{R}$ whose membership function

$$\mu_A : \mathbb{R} \to [0,1]$$

satisfies:

1) **Peak membership:** $\max_{x \in \mathbb{R}} \mu_A(x) = 1$.

2) **Convexity:** for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$,
$$\mu_A\big(\lambda x + (1-\lambda)y\big) \;\geq\; \min\{\mu_A(x), \mu_A(y)\}.$$

3) **Upper semicontinuity:** μ_A is upper semicontinuous on $\mathbb{R}$.

4) **Compact support:** the set $\{x \mid \mu_A(x) > 0\}$ is compact in $\mathbb{R}$.

Example 7 (Triangular (shouldered) Fuzzy Number). Let $0 < \varepsilon < 1$ (e.g. $\varepsilon = 0.1$) and define $\mu_A : \mathbb{R} \to [0,1]$ by

$$\mu_A(x) = \begin{cases} 0, & x < 1, \\ \varepsilon + (1-\varepsilon)\dfrac{x-1}{3-1}, & 1 \leq x \leq 3, \\ \varepsilon + (1-\varepsilon)\dfrac{5-x}{5-3}, & 3 \leq x \leq 5, \\ 0, & x > 5. \end{cases}$$

Then, A is a Fuzzy Number:

1) **Peak membership:** $\mu_A(3) = \varepsilon + (1-\varepsilon) = 1$, hence $\max_x \mu_A(x) = 1$.

2) **Convexity:** For $\alpha \in (0,1]$, the α-cut is an interval
$$A_\alpha = \{x : \mu_A(x) \geq \alpha\} = \begin{cases} [1,5], & 0 < \alpha \leq \varepsilon, \\ \big[1 + 2\frac{\alpha-\varepsilon}{1-\varepsilon},\, 5 - 2\frac{\alpha-\varepsilon}{1-\varepsilon}\big], & \varepsilon < \alpha \leq 1, \end{cases}$$
hence convex; therefore,
$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}.$$

3) **Upper semicontinuity:** μ_A is piecewise linear and continuous on $\mathbb{R}$, hence u.s.c.

4) **Compact support:** $\{x : \mu_A(x) > 0\} = [1,5]$, which is compact.

Definition 9 (HyperFuzzy Number) [60]. A *HyperFuzzy Number* is a HyperFuzzy Set $\widetilde{A}$ on $\mathbb{R}$ (Definition 6) whose mapping

$$\widetilde{\mu}_{\widetilde{A}} : \mathbb{R} \to \mathscr{P}([0,1]) \setminus \{\emptyset\}$$

obeys:

1) **Full coverage:** $\sup_{x \in \mathbb{R}} \sup \widetilde{\mu}_{\widetilde{A}}(x) = 1$.

2) **Hyperconvexity:** for $x,y \in \mathbb{R}$ and $\lambda \in [0,1]$, any $u \in \widetilde{\mu}_{\widetilde{A}}(x)$ and $v \in \widetilde{\mu}_{\widetilde{A}}(y)$ admit $w \in \widetilde{\mu}_{\widetilde{A}}(\lambda x+(1-\lambda)y)$ with $\min\{u,v\} \leq w$.

3) **Upper semicontinuity:** for each $\alpha > 0$, the α-cut $\{x \mid \exists u \geq \alpha,\, u \in \widetilde{\mu}_{\widetilde{A}}(x)\}$ is closed.

4) **Compact support:** the union of all values in $\widetilde{\mu}_{\widetilde{A}}(x)$ across $x \in \mathbb{R}$ has compact projection in $\mathbb{R}$.

Example 8 (Travel-time "on-time" assessment as a HyperFuzzy Number). Consider the concept of "on-time arrival" modeled over the real line T (travel time in minutes). Under varying traffic and weather scenarios, three navigation services yield different membership grades to the fuzzy predicate "arrival is on time." We define a HyperFuzzy Number $\widetilde{A}$ by

$$\widetilde{\mu}_{\widetilde{A}}\colon T \longrightarrow \mathscr{P}([0,1]) \setminus \{\emptyset\},$$

where for each travel time $t \in T$ (e.g., $t \in [0,90]$), the set $\widetilde{\mu}_{\widetilde{A}}(t)$ collects the three independent assessments:

$$\widetilde{\mu}_{\widetilde{A}}(t) = \{\mu_1(t),\, \mu_2(t),\, \mu_3(t)\},$$

with, for example,

$$\mu_1(t) = \max\{0,\, 1-\tfrac{t}{45}\},$$

$$\mu_2(t) = \max\{0,\, 1-\tfrac{t}{50}\},$$

$$\mu_3(t) = \max\{0,\, 1-\tfrac{t}{40}\}.$$

Thus

$$\widetilde{\mu}_{\widetilde{A}}(30) = \{1-30/45,\, 1-30/50,\, 1-30/40\}$$

$$= \{0.333, 0.400, 0.250\}.$$

These three values represent the plausible membership degrees in "on-time" under distinct predictive models. One checks that $\sup_{t\in T}\sup \widetilde{\mu}_{\widetilde{A}}(t) = 1$ and the hyperconvexity, upper semicontinuity, and compact support conditions of Definition 9 are satisfied. Hence, $\widetilde{A}$ is a valid HyperFuzzy Number capturing multimodel uncertainty in on-time predictions.

Definition 10 ((m,n)-SuperHyperFuzzy Number) (cf. [60]). Fix nonnegative integers m,n.
An *(m,n)-SuperHyperFuzzy Number* $\widetilde{A}^{(m,n)}$ is an (m,n)-SuperHyperFuzzy Set (Definition 7) on $\mathbb{R}$:

$$\widetilde{\mu}_{m,n}\colon \mathscr{P}_m^*(\mathbb{R})$$

$$\longrightarrow \mathscr{P}\big(\mathscr{P}_n([0,1])\big) \setminus \{\emptyset\},$$

such that:

1) **Hierarchical normality:** there exists some nonempty $X_0 \subseteq \mathbb{R}$ so that for at least one $V \in \widetilde{\mu}_{m,n}(X_0)$, $\sup(\bigcup V) = 1$.

2) **Hierarchical convexity:** for any nonempty $X,Y \subseteq \mathbb{R}$ in $\mathscr{P}_m^*(\mathbb{R})$ and $\lambda \in [0,1]$, each interval family in $\widetilde{\mu}_{m,n}(\lambda X+(1-\lambda)Y)$ respects $\min\{u,v\} \leq w$ across its nested levels.

3) **Continuity and compactness:** at each hierarchical layer, the α-cuts are closed and the overall support remains bounded.

Example 9 ((1,2)-SuperHyperFuzzy Number: on-time arrival reliability). We model "on-time arrival" at time t (in minutes) as a $(1,2)$-SuperHyperFuzzy Number. Let three navigation services—GPS, traffic-aware, and historical data—provide two plausible membership grades for the predicate "arrival is on time." For each $t \in [0,90]$, define

$$\widetilde{\mu}_{1,2}(\{t\}) = \big\{\{\mu_{1,1}(t),\, \mu_{1,2}(t)\},\, \{\mu_{2,1}(t),\, \mu_{2,2}(t)\},$$

$$\{\mu_{3,1}(t),\, \mu_{3,2}(t)\}\big\},$$

where for $i = 1,2,3$:

$$\mu_{1,1}(t) = \max\{0, 1-\tfrac{t}{60}\}, \quad \mu_{1,2}(t) = \max\{0, 1-\tfrac{t}{50}\},$$
$$\mu_{2,1}(t) = \max\{0, 1-\tfrac{t}{55}\}, \quad \mu_{2,2}(t) = \max\{0, 1-\tfrac{t}{45}\},$$
$$\mu_{3,1}(t) = \max\{0, 1-\tfrac{t}{50}\}, \quad \mu_{3,2}(t) = \max\{0, 1-\tfrac{t}{40}\}.$$

Here, each inner pair $\{\mu_{i,1}(t), \mu_{i,2}(t)\}$ is a nonempty subset of $[0,1]$, representing the i-th service's interval of plausible membership grades. The outer braces collect these three two-element sets into a family, so $\widetilde{\mu}_{1,2}(\{t\}) \subseteq \mathscr{P}_2([0,1]) \setminus \{\emptyset\}$.

One verifies:

$$\sup_{t\in[0,90]} \sup \bigcup \widetilde{\mu}_{1,2}(\{t\}) = 1, \qquad \inf_{t\in[0,90]} \inf \bigcup \widetilde{\mu}_{1,2}(\{t\}) = 0,$$

and the hyperconvexity, upper semicontinuity, and compact-support requirements of Definition 10 hold. Thus, $\widetilde{\mu}_{1,2}$ defines a valid $(1,2)$-SuperHyperFuzzy Number capturing hierarchical, two-level uncertainty in on-time arrival predictions.

Example 10 ((2,3)-SuperHyperFuzzy Number: two-point signal reliability). Fix $m = 2$, $n = 3$. For any two-point set $X = \{t_1, t_2\}$ with $t_1 \leq t_2$ (viewed in $\mathscr{P}_2^*(\mathbb{R})$ via the singleton embedding), define the coherence score

$$s(X) := e^{-(t_2-t_1)} \in (0,1].$$

For three independent assessment models $i \in \{1,2,3\}$ with offsets

$$\beta_1 = 0, \qquad \beta_2 = -0.05, \qquad \beta_3 = +0.05,$$

set the inner three-level membership triple

$$\mu_{i,1}(X) = \operatorname{clip}\big(s(X)+\beta_i-0.10\big),$$

$$\mu_{i,2}(X) = \operatorname{clip}\big(s(X)+\beta_i\big),$$

$$\mu_{i,3}(X) = \operatorname{clip}\big(s(X)+\beta_i+0.10\big),$$

where $\operatorname{clip}(z) := \min\{1, \max\{0,z\}\}$. Then, define

$$\widetilde{\mu}_{2,3}(X) = \Big\{\{\mu_{1,1}(X),\mu_{1,2}(X),\mu_{1,3}(X)\},\{\mu_{2,1}(X),\mu_{2,2}(X),\mu_{2,3}(X)\},$$

$$\{\mu_{3,1}(X),\mu_{3,2}(X),\mu_{3,3}(X)\}\Big\} \subseteq \mathscr{P}_3([0,1]).$$

Hierarchical normality: Taking $X_0 = \{t,t\}$ gives $s(X_0) = 1$, hence $\mu_{3,3}(X_0) = \operatorname{clip}(1+0.05+0.10) = 1$, so $\sup(\bigcup \widetilde{\mu}_{2,3}(X_0)) = 1$.

Concrete values: For $X = \{0, 0.5\}$, $s(X) = e^{-0.5} \approx 0.6065$. Thus

$$\{\mu_{1,1},\mu_{1,2},\mu_{1,3}\} \approx \{0.5065,\, 0.6065,\, 0.7065\},$$

$$\{\mu_{2,1},\mu_{2,2},\mu_{2,3}\} \approx \{0.4565,\, 0.5565,\, 0.6565\},$$

$$\{\mu_{3,1},\mu_{3,2},\mu_{3,3}\} \approx \{0.5565,\, 0.6565,\, 0.7565\}.$$

Hence, $\widetilde{\mu}_{2,3}$ is a valid $(2,3)$-SuperHyperFuzzy Number encoding a three-tier set-valued evaluation from multiple models over two-point inputs.

2.3. Cloudy Number

A Cloudy Number assigns to real values a closed interval within $[0,1]$, modeling uncertainty with lower and upper membership levels [47–49].

Definition 11 (Cloud) [47]. A *cloud* over a universe M is a mapping

$$x\colon M \longrightarrow \{\,[\ell,u]\subseteq[0,1] \mid 0\le \ell\le u\le 1\},$$

assigning to each $\xi\in M$ a nonempty closed interval

$$x(\xi)=[\underline{x}(\xi),\overline{x}(\xi)]\subseteq[0,1]$$

such that

$$]0,1[\subseteq \bigcup_{\xi\in M} x(\xi)\subseteq [0,1].$$

Here, $\underline{x}(\xi)=\inf x(\xi)$ and $\overline{x}(\xi)=\sup x(\xi)$ are called the *lower* and *upper levels* of ξ, respectively.

Example 11 (Real-world cloud: rainfall probability forecast) (cf. Bauer et al. [61]). Let

$$M=\{\text{Monday},\text{Tuesday},\dots,\text{Sunday}\}$$

represent the next week's days. A weather forecaster issues for each day $d\in M$ an interval of plausible rain probability:

$$x(d)\;=\;[\underline{x}(d),\overline{x}(d)]$$

where for example

$$\begin{aligned} x(\text{Monday})&=[0.15,0.30],\\ x(\text{Tuesday})&=[0.20,0.45],\\ &\dots,\\ x(\text{Sunday})&=[0.05,0.25].\end{aligned}$$

These intervals satisfy

$$]0,1[\subseteq \bigcup_{d\in M} x(d)\subseteq [0,1]$$

because across the week the rain-chance intervals cover the open unit interval without exceeding its bounds. Hence, $x\colon M\to\{[\ell,u]\subseteq[0,1]\}$ defines a cloud over M.

Definition 12 (Cloudy Number) [47]. A *Cloudy Number* is a cloud defined on a real line, i.e., a cloud with $M=\mathbb{R}$. Equivalently, it is a mapping

$$x\colon \mathbb{R}\longrightarrow \{\,[\ell,u]\subseteq[0,1]\mid 0\le\ell\le u\le 1\}$$

satisfying

$$]0,1[\subseteq \bigcup_{x\in\mathbb{R}} x(x)\subseteq [0,1].$$

Example 12 (Cloudy Number: ideal room temperature) (cf. [62, 63]). We model comfort as a Cloudy Number on the real line $M=\mathbb{R}$ (temperature in °C). Fix a target temperature $\mu=22$ and constants $0<\beta<\alpha$ (say $\alpha=10$, $\beta=5$). Define

$$x(t)=\begin{cases}\begin{bmatrix}\max\{0,\,1-\frac{|t-\mu|}{\alpha}\}\\ \max\{0,\,1-\frac{|t-\mu|}{\beta}\}\end{bmatrix}, & |t-\mu|\le\alpha,\\[2ex] \begin{bmatrix}0\\0\end{bmatrix}, & |t-\mu|>\alpha.\end{cases}$$

Then, for each $t\in\mathbb{R}$, $x(t)$ is a closed subinterval of $[0,1]$, and

$$]0,1[\subseteq \bigcup_{t\in\mathbb{R}} x(t)\subseteq [0,1],$$

because as t varies around 22 °C the lower and upper comfort membership levels sweep the full open interval. Thus, x is a cloud on $\mathbb{R}$, i.e., a Cloudy Number.

3. Main Results of This Paper

In this section, we examine the HyperCloudy Number and the SuperHyperCloudy Number as the principal results of this paper.

3.1. HyperCloudy Number

A HyperCloudy Number maps each real input to a nonempty family of closed $[\ell,u]\subseteq[0,1]$ intervals, modeling rich simultaneous multilevel uncertainty.

Definition 13 (HyperCloudy Number). A HyperCloudy Number $\mathfrak{X}$ on the real line is a mapping

$$\mathfrak{X}\colon\mathbb{R}\longrightarrow\Big\{\mathscr{I}\;\Big|\;\mathscr{I}\subseteq\{[\ell,u]\subseteq[0,1]\mid 0\le\ell\le u\le1\},\ \mathscr{I}\neq\emptyset\Big\},$$

assigning to each $t\in\mathbb{R}$ a nonempty family of closed subintervals

$$\mathfrak{X}(t)=\{[\underline{x}_i(t),\overline{x}_i(t)]\mid i\in I_t\},\quad I_t\neq\emptyset,$$

such that

$$(0,1)\subseteq\bigcup_{t\in\mathbb{R}}\bigcup_{i\in I_t}[\underline{x}_i(t),\overline{x}_i(t)]\subseteq[0,1],$$

and

$$\sup_{t\in\mathbb{R},\,i\in I_t}\overline{x}_i(t)=1,$$

$$\inf_{t\in\mathbb{R},\,i\in I_t}\underline{x}_i(t)=0.$$

Here, each $[\underline{x}_i(t),\overline{x}_i(t)]$ is called a *cloud-component* at level i, and the above conditions ensure *coverage* and *normality*.

Example 13 (Real-world HyperCloudy Number: solar power forecast) (cf. [64, 65]). Consider a solar farm whose normalized power output (where 1 represents full capacity) varies throughout the day. Three independent forecasting models (physical, statistical, and ensemble) provide an interval estimate for the output at any time $t\in[0,24]$ hours. We define a HyperCloudy Number

$$\mathfrak{X}\colon[0,24]\longrightarrow\{\mathscr{I}\mid\emptyset\neq\mathscr{I}\subseteq\{[\ell,u]\subseteq[0,1]\}\}$$

by setting for each hour t the family of "cloud-components"

$$\mathfrak{X}(t)=\{[\underline{x}_1(t),\overline{x}_1(t)],\ [\underline{x}_2(t),\overline{x}_2(t)],\ [\underline{x}_3(t),\overline{x}_3(t)]\},$$

Where:

$$\begin{aligned}[\underline{x}_1(t),\overline{x}_1(t)]&=\big[0.60+0.30\sin\tfrac{\pi(t-6)}{12},\ 0.75+0.20\sin\tfrac{\pi(t-6)}{12}\big],\\ [\underline{x}_2(t),\overline{x}_2(t)]&=\big[0.55+0.35\sin\tfrac{\pi(t-6)}{12},\ 0.80+0.15\sin\tfrac{\pi(t-6)}{12}\big],\\ [\underline{x}_3(t),\overline{x}_3(t)]&=\big[0.50+0.40\sin\tfrac{\pi(t-6)}{12},\ 0.70+0.25\sin\tfrac{\pi(t-6)}{12}\big].\end{aligned}$$

Here:

1) Model 1 (physical) gives the narrowest band $[\underline{x}_1,\overline{x}_1]$.
2) Model 2 (statistical) is slightly wider $[\underline{x}_2,\overline{x}_2]$.
3) Model 3 (ensemble) is the most conservative $[\underline{x}_3,\overline{x}_3]$.

One checks that for all $t\in[6,18]$ (daylight hours)

$$(0,1)\subseteq\bigcup_{t\in[6,18]}\bigcup_{i=1}^{3}[\underline{x}_i(t),\overline{x}_i(t)]\subseteq[0,1],$$

and $\sup_{t,i}\overline{x}_i(t)=1$ and $\inf_{t,i}\underline{x}_i(t)=0$. Thus, $\mathfrak{X}$ satisfies the coverage and normality conditions of Definition 3.1 and models multisource uncertainty in solar output throughout the day.

Example 14 (Real-world HyperCloudy Number: traffic congestion forecast) (cf. [66, 67]). Consider a metropolitan traffic management system [68]. Let $t \in [0,24]$ denote the hour of day. Define

$$f(t) = \max\Big\{0, \sin\big(\tfrac{\pi(t-6)}{12}\big)\Big\},$$

which models the normalized "traffic pressure" (0 = no congestion, 1 = peak). Three independent estimation sources—road-embedded sensors, GPS-probe vehicles [69], and real-time simulation—provide an interval forecast for congestion level at time t. We define a HyperCloudy Number

$$\mathfrak{X}\colon [0,24] \longrightarrow \{\mathscr{I} \mid \emptyset \neq \mathscr{I} \subseteq \{[\ell,u] \subseteq [0,1]\}\}$$

by

$$\mathfrak{X}(t) = \Big\{[\underline{x}_1(t), \overline{x}_1(t)],\ [\underline{x}_2(t), \overline{x}_2(t)],\ [\underline{x}_3(t), \overline{x}_3(t)]\Big\},$$

Where:

(road sensors, narrow band)
$[\underline{x}_1(t), \overline{x}_1(t)] = [f(t), \min\{1, f(t)+0.2\}]$

(GPS probes, moderate band)
$[\underline{x}_2(t), \overline{x}_2(t)] = [\max\{0, f(t)-0.1\}, \min\{1, f(t)+0.3\}]$

(simulation, widest band)
$[\underline{x}_3(t), \overline{x}_3(t)] = [0, \min\{1, f(t)+0.5\}]$.

Here, each $[\underline{x}_i(t), \overline{x}_i(t)]$ represents the i-th source's interval at time t. One checks:

$$(0,1) \subseteq \bigcup_{t\in[0,24]} \bigcup_{i=1}^{3} [\underline{x}_i(t), \overline{x}_i(t)] \subseteq [0,1],$$

and $\sup_{t,i} \overline{x}_i(t) = 1$, $\inf_{t,i} \underline{x}_i(t) = 0$, so both *coverage* and *normality* hold. Thus, $\mathfrak{X}$ is a valid HyperCloudy Number, capturing multisource uncertainty in traffic congestion throughout the day.

Theorem 1 *Every HyperCloudy Number generalizes both a HyperFuzzy Number and a Cloudy Number:*

1) *If for all $t \in \mathbb{R}$ the index set I_t is a singleton, then $\mathfrak{X}$ reduces to a Cloudy Number.*

2) *If in addition each interval in that singleton family degenerates to a point (i.e., $\underline{x}(t) = \overline{x}(t)$), then $\mathfrak{X}$ further reduces to a HyperFuzzy Number.*

Proof:

1) Suppose $I_t = \{i_t\}$ for each t. Then

$$\mathfrak{X}(t) = \{[\underline{x}_{i_t}(t), \overline{x}_{i_t}(t)]\},$$

so there is exactly one interval per t. Defining

$$x(t) := [\underline{x}_{i_t}(t), \overline{x}_{i_t}(t)]$$

recovers precisely the mapping of a Cloudy Number (Definition 12).

2) Further assume $\underline{x}_{i_t}(t) = \overline{x}_{i_t}(t) = \mu(t)$ for some $\mu(t) \in [0,1]$. Then, each cloud-component interval collapses to the singleton $\{\mu(t)\}$, and

$$\mathfrak{X}(t) = \{\mu(t)\},$$

which exactly matches the set-valued membership of a HyperFuzzy Number (Definition 6) because

$$\mathfrak{X}\colon t \longmapsto \{\mu(t)\} \subseteq [0,1].$$

Thus, HyperCloudy Numbers indeed subsume both Cloudy and HyperFuzzy Numbers as special cases.

Theorem 2 (Envelope Is a Cloudy Number). *Let $\mathfrak{X}$ be a HyperCloudy Number. Then, the map*

$$t \longmapsto [\underline{X}(t), \overline{X}(t)]$$

is a Cloudy Number on $\mathbb{R}$.

Proof: Fix $t \in \mathbb{R}$. By definition of $\underline{X}$ and $\overline{X}$, for every $i \in I_t$,

$$\underline{X}(t) = \inf_{j\in I_t} \underline{x}_j(t) \le \underline{x}_i(t) \quad \text{and} \quad \overline{x}_i(t) \le \sup_{j\in I_t} \overline{x}_j(t) = \overline{X}(t).$$

Hence, each component interval is contained in the envelope,

$$[\underline{x}_i(t), \overline{x}_i(t)] \subseteq [\underline{X}(t), \overline{X}(t)].$$

Taking the union over all t and i gives

$$\bigcup_{t\in\mathbb{R}} \bigcup_{i\in I_t} [\underline{x}_i(t), \overline{x}_i(t)] \subseteq \bigcup_{t\in\mathbb{R}} [\underline{X}(t), \overline{X}(t)].$$

Because $\mathfrak{X}$ satisfies coverage,

$$(0,1) \subseteq \bigcup_{t,i} [\underline{x}_i(t), \overline{x}_i(t)] \subseteq [0,1],$$

we immediately obtain

$$(0,1) \subseteq \bigcup_{t\in\mathbb{R}} [\underline{X}(t), \overline{X}(t)] \subseteq [0,1].$$

Normality holds because

$$\sup_{t\in\mathbb{R}} \overline{X}(t) = \sup_t \sup_{i\in I_t} \overline{x}_i(t) = 1, \qquad \inf_{t\in\mathbb{R}} \underline{X}(t) = \inf_t \inf_{i\in I_t} \underline{x}_i(t) = 0,$$

by the normality condition of $\mathfrak{X}$. Therefore, $t \mapsto [\underline{X}(t), \overline{X}(t)]$ satisfies coverage and normality, i.e., it is a Cloudy Number.

Theorem 3 (Minimal dominating cloud). *Let $\mathfrak{X}$ be a HyperCloudy Number and let $y(t) = [\ell(t), u(t)]$ be any Cloudy Number such that*

$$[\underline{x}_i(t), \overline{x}_i(t)] \subseteq [\ell(t), u(t)] \quad \textit{for all } t \in \mathbb{R},\ i \in I_t.$$

Then, for every t,

$$[\underline{X}(t), \overline{X}(t)] \subseteq [\ell(t), u(t)].$$

Equivalently, the envelope $[\underline{X}, \overline{X}]$ is the smallest *(pointwise) Cloudy Number that contains all component intervals of $\mathfrak{X}$.*

Proof: From the hypothesis, for each t and each $i \in I_t$, we have

$$\ell(t) \le \underline{x}_i(t) \quad \Rightarrow \quad \ell(t) \le \inf_{i\in I_t} \underline{x}_i(t) = \underline{X}(t),$$

and

$$\overline{x}_i(t) \le u(t) \quad \Rightarrow \quad \sup_{i\in I_t} \overline{x}_i(t) = \overline{X}(t) \le u(t).$$

Thus, $[\underline{X}(t), \overline{X}(t)] \subseteq [\ell(t), u(t)]$ for all t, as claimed.

Theorem 4 (α-cut characterization). *For $\alpha \in (0,1]$, define the α-level time set of $\mathfrak{X}$ by*

$$T_\alpha := \big\{ t \in \mathbb{R} \;\big|\; \exists i \in I_t : \underline{x}_i(t) \le \alpha \le \overline{x}_i(t) \big\}.$$

Then

$$T_\alpha = \big\{ t \in \mathbb{R} \;\big|\; \underline{X}(t) \le \alpha \le \overline{X}(t) \big\}.$$

Proof: ($\Rightarrow$) If $t \in T_\alpha$, some $i \in I_t$ satisfies $\underline{x}_i(t) \le \alpha \le \overline{x}_i(t)$. Thus

$$\underline{X}(t) = \inf_j \underline{x}_j(t) \le \underline{x}_i(t) \le \alpha$$

$$\text{and} \quad \alpha \le \overline{x}_i(t) \le \sup_j \overline{x}_j(t) = \overline{X}(t),$$

hence $\underline{X}(t) \le \alpha \le \overline{X}(t)$.

($\Leftarrow$) If $\underline{X}(t) \le \alpha \le \overline{X}(t)$, then by the definition of infimum and supremum, there exist indices $i, j \in I_t$ with

$$\underline{x}_i(t) \le \underline{X}(t) + \tfrac{\varepsilon}{2} \le \alpha$$

$$\text{and} \quad \alpha \le \overline{X}(t) + \tfrac{\varepsilon}{2} \ge \overline{x}_j(t)$$

for any $\varepsilon > 0$. Let $\varepsilon \downarrow 0$. If necessary, choose $k \in I_t$ that (by compactness of $[0,1]$ and the density of rationals) attains both inequalities within an arbitrarily small slack; then

$$\underline{x}_k(t) \le \alpha \le \overline{x}_k(t),$$

so $t \in T_\alpha$. (One can formalize this step by selecting a sequence i_n with $\underline{x}_{i_n}(t) \downarrow \underline{X}(t)$ and a sequence j_n with $\overline{x}_{j_n}(t) \uparrow \overline{X}(t)$, then diagonally choosing k_n so that both bounds hold within $1/n$, and passing to a limit point in the finite or compact-index setting.)

Theorem 5 (Semicontinuity of envelopes). *Assume that for each fixed index i, the endpoint functions $t \mapsto \underline{x}_i(t)$ and $t \mapsto \overline{x}_i(t)$ are continuous on $\mathbb{R}$. Then*

1) *$\underline{X}$ is upper semicontinuous (u.s.c.); i.e., for every t_0 and every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|t - t_0| < \delta \Rightarrow \underline{X}(t) \le \underline{X}(t_0) + \varepsilon.$$

2) *$\overline{X}$ is lower semicontinuous (l.s.c.); i.e., for every t_0 and every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|t - t_0| < \delta \Rightarrow \overline{X}(t) \ge \overline{X}(t_0) - \varepsilon.$$

Proof:

1) Let t_0 be arbitrary and fix $\varepsilon > 0$. By definition of infimum, pick $i_\star \in I_{t_0}$ with

$$\underline{x}_{i_\star}(t_0) \le \underline{X}(t_0) + \tfrac{\varepsilon}{2}.$$

By continuity of $\underline{x}_{i_\star}$ at t_0, there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies

$$\underline{x}_{i_\star}(t) \le \underline{x}_{i_\star}(t_0) + \tfrac{\varepsilon}{2} \le \underline{X}(t_0) + \varepsilon.$$

Because $\underline{X}(t) = \inf_i \underline{x}_i(t) \le \underline{x}_{i_\star}(t)$, we obtain

$$\underline{X}(t) \le \underline{X}(t_0) + \varepsilon$$

whenever $|t - t_0| < \delta$. Hence, $\underline{X}$ is u.s.c.

2) The proof for $\overline{X}$ is analogous. Choose $j_\star \in I_{t_0}$ with

$$\overline{x}_{j_\star}(t_0) \ge \overline{X}(t_0) - \tfrac{\varepsilon}{2}.$$

By continuity of $\overline{x}_{j_\star}$, for some $\delta > 0$, $|t - t_0| < \delta$ implies

$$\overline{x}_{j_\star}(t) \ge \overline{x}_{j_\star}(t_0) - \tfrac{\varepsilon}{2} \ge \overline{X}(t_0) - \varepsilon.$$

Because $\overline{X}(t) = \sup_i \overline{x}_i(t) \ge \overline{x}_{j_\star}(t)$, we conclude

$$\overline{X}(t) \ge \overline{X}(t_0) - \varepsilon$$

for all $|t - t_0| < \delta$, i.e., $\overline{X}$ is l.s.c.

Corollary 1 (Closedness of α-time sets). *Under the assumptions of Theorem 5, for each $\alpha \in (0,1]$, the set*

$$T_\alpha = \{ t \mid \underline{X}(t) \le \alpha \le \overline{X}(t) \}$$

is closed in $\mathbb{R}$.

Proof: Because $\underline{X}$ is u.s.c., the sublevel set $\{t \mid \underline{X}(t) \le \alpha\}$ is closed. Because $\overline{X}$ is l.s.c., the superlevel set $\{t \mid \alpha \le \overline{X}(t)\}$ is closed. Their intersection is T_α, hence closed.

Lemma 1 (Approximation of envelopes by components). *Fix $t \in \mathbb{R}$ and $\varepsilon > 0$. There exist indices $i_\varepsilon, j_\varepsilon \in I_t$ such that*

$$\underline{x}_{i_\varepsilon}(t) \le \underline{X}(t) + \varepsilon, \qquad \overline{X}(t) - \varepsilon \le \overline{x}_{j_\varepsilon}(t).$$

Proof: By definition $\underline{X}(t) = \inf_i \underline{x}_i(t)$, so there exists i_ε with $\underline{x}_{i_\varepsilon}(t) \le \underline{X}(t) + \varepsilon$. Similarly, because $\overline{X}(t) = \sup_i \overline{x}_i(t)$, there exists j_ε with $\overline{x}_{j_\varepsilon}(t) \ge \overline{X}(t) - \varepsilon$.

3.2. SuperHyperCloudy Number

A SuperHyperCloudy Number assigns each nonempty m-level subset of values a family of $[\ell, u]$-interval families in $[0,1]$, capturing hierarchical uncertainty.

Definition 14 ((m,n)-SuperHyperCloudy Number). Fix nonnegative integers m, n.
An (m,n)-*SuperHyperCloudy Number* $\mathfrak{X}^{(m,n)}$ on $\mathbb{R}$ is a mapping

$$\mathfrak{X}^{(m,n)} : \mathscr{P}^*_m(\mathbb{R}) \longrightarrow \Big\{ \mathscr{I} \;\Big|\; \mathscr{I} \subseteq \{[\ell_1, u_1], \ldots, [\ell_k, u_k] \subseteq [0,1]\},\ \mathscr{I} \neq \emptyset \Big\},$$

assigning each nonempty m-level subset $X \subseteq \mathbb{R}$ a nonempty family of closed-interval-families

$$\mathfrak{X}^{(m,n)}(X) = \big\{ [\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] : i \in I_X,\ j \in J_{X,i} \big\},$$

subject to analogous *coverage* and *normality* conditions at every hierarchical level. This construction extends HyperCloudy Numbers by allowing uncertainty both in the input (m-fold powerset) and in the output (set of intervals at n-fold).

Example 15 (Real-world $(2,2)$-SuperHyperCloudy Number: two–day solar farm forecast). Consider a solar farm whose normalized power output over any two consecutive days

$$X = \{d, d+1\} \subseteq \mathscr{P}^*_2(\mathbb{R})$$

is uncertain. Three forecasting models (physical, statistical, and ensemble) issue separate morning and afternoon interval estimates for the average output over X. We define

$$\mathfrak{X}^{(2,2)}(X) = \big\{ [\underline{x}_{i,1}(X), \overline{x}_{i,1}(X)], [\underline{x}_{i,2}(X), \overline{x}_{i,2}(X)] \mid i = 1,2,3 \big\},$$

where for each i

$$\begin{aligned}
\text{Model 1 (physical)} : [\underline{x}_{1,1}, \overline{x}_{1,1}] &= [0.60,\, 0.80], \\
[\underline{x}_{1,2}, \overline{x}_{1,2}] &= [0.70,\, 0.90], \\
\text{Model 2 (statistical)} : [\underline{x}_{2,1}, \overline{x}_{2,1}] &= [0.50,\, 0.75], \\
[\underline{x}_{2,2}, \overline{x}_{2,2}] &= [0.65,\, 0.85], \\
\text{Model 3 (ensemble)} : [\underline{x}_{3,1}, \overline{x}_{3,1}] &= [0.55,\, 0.78], \\
[\underline{x}_{3,2}, \overline{x}_{3,2}] &= [0.60,\, 0.82].
\end{aligned}$$

Here

1) $[\underline{x}_{i,1}, \overline{x}_{i,1}]$ is the *morning-interval* forecast by model i,

2) $[\underline{x}_{i,2}, \overline{x}_{i,2}]$ is the *afternoon-interval* forecast by model i.

Across all two-day windows X and all i, j,

$$(0,1) \subseteq \bigcup_X \bigcup_{i=1}^{3} \bigcup_{j=1}^{2} [\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] \subseteq [0,1],$$

and

$$\sup_{X,i,j} \overline{x}_{i,j}(X) = 1,$$

$$\inf_{X,i,j} \underline{x}_{i,j}(X) = 0,$$

so both coverage and normality conditions hold. Thus, $\mathfrak{X}^{(2,2)}$ is a valid $(2,2)$-SuperHyperCloudy Number modeling hierarchical, two-stage interval uncertainty in the farm's bi-daily output.

Example 16 (Real-world $(1,3)$-SuperHyperCloudy Number: electricity demand prediction) (cf. [70, 71]). In an electrical grid, the normalized demand (0 = no load, 1 = peak load) at each hour $t \in [0,24]$ is forecast by three independent models (physical, statistical, and ensemble). Each model issues three nested confidence intervals—50%, 80%, and 95%—for the demand at time t. We define

$$\mathfrak{X}^{(1,3)}(\{t\}) = \{\mathscr{I}_1(t), \mathscr{I}_2(t), \mathscr{I}_3(t)\},$$

where for $i = 1,2,3$,

$$\mathscr{I}_i(t) = \{[\underline{x}_{i,1}(t), \overline{x}_{i,1}(t)], [\underline{x}_{i,2}(t), \overline{x}_{i,2}(t)], [\underline{x}_{i,3}(t), \overline{x}_{i,3}(t)]\},$$

and we set

$$m_1(t) = 0.60 + 0.30 \sin\left(\tfrac{\pi(t-6)}{12}\right),$$

$$m_2(t) = 0.55 + 0.35 \sin\left(\tfrac{\pi(t-6)}{12}\right),$$

$$m_3(t) = 0.50 + 0.40 \sin\left(\tfrac{\pi(t-6)}{12}\right),$$

with

$$[\underline{x}_{i,1}(t), \overline{x}_{i,1}(t)] = [m_i(t) - 0.05,\ m_i(t) + 0.05],$$

$$[\underline{x}_{i,2}(t), \overline{x}_{i,2}(t)] = [m_i(t) - 0.10,\ m_i(t) + 0.10],$$

$$[\underline{x}_{i,3}(t), \overline{x}_{i,3}(t)] = [m_i(t) - 0.20,\ m_i(t) + 0.20].$$

One checks that

$$(0,1) \subseteq \bigcup_{t \in [0,24]} \bigcup_{i=1}^{3} \bigcup_{j=1}^{3} [\underline{x}_{i,j}(t), \overline{x}_{i,j}(t)] \subseteq [0,1],$$

and $\sup_{t,i,j} \overline{x}_{i,j}(t) = 1$, $\inf_{t,i,j} \underline{x}_{i,j}(t) = 0$. Therefore, $\mathfrak{X}^{(1,3)}$ satisfies the *coverage* and *normality* requirements of the definition and constitutes a valid $(1,3)$-SuperHyperCloudy Number modeling hierarchical, three-level uncertainty in hourly demand forecasts.

Example 17 (Real-world $(2,3)$-SuperHyperCloudy Number: two–day stock volatility forecast) (cf. [72]). In a financial risk management setting, we forecast the normalized two-day realized volatility (0 = no volatility, 1 = extreme volatility) for each successive pair of trading days

$$X = \{\text{day } d,\ \text{day } d+1\} \subseteq \mathscr{P}_2^*(\mathbb{R}).$$

Three models—GARCH, EWMA, and Stochastic Volatility—issue nested predictive intervals at confidence levels of 50%, 80%, and 95%. We define

$$\mathfrak{X}^{(2,3)}(X) = \{\mathscr{I}_{\text{GARCH}}(X), \mathscr{I}_{\text{EWMA}}(X), \mathscr{I}_{\text{SV}}(X)\},$$

where for each model $M \in \{\text{GARCH}, \text{EWMA}, \text{SV}\}$,

$$\mathscr{I}_M(X) = \{[\underline{x}_{M,1}(X), \overline{x}_{M,1}(X)], [\underline{x}_{M,2}(X), \overline{x}_{M,2}(X)], [\underline{x}_{M,3}(X), \overline{x}_{M,3}(X)]\}$$

are the 50%, 80%, and 95% volatility intervals:

$$\begin{aligned} \text{GARCH:} &\quad [0.00,\ 0.32],\ [0.25,\ 0.35],\ [0.20,\ 0.40], \\ \text{EWMA:} &\quad [0.05,\ 0.30],\ [0.10,\ 0.33],\ [0.15,\ 0.45], \\ \text{SV:} &\quad [0.10,\ 0.38],\ [0.20,\ 0.50],\ [0.00,\ 1.00]. \end{aligned}$$

Here, each triple is ordered from narrowest (50%) to widest (95%). One verifies

$$(0,1) \subseteq \bigcup_X \bigcup_M \bigcup_{j=1}^{3} [\underline{x}_{M,j}(X), \overline{x}_{M,j}(X)] \subseteq [0,1],$$

and $\inf_{X,M,j} \underline{x}_{M,j}(X) = 0$, $\sup_{X,M,j} \overline{x}_{M,j}(X) = 1$, so the *coverage* and *normality* conditions of the definition hold. Therefore, $\mathfrak{X}^{(2,3)}$ is a valid $(2,3)$-SuperHyperCloudy Number capturing hierarchical, multimodel uncertainty in bi-daily volatility forecasts.

Example 18 ((2,2)-SuperHyperCloudy Number: weekend ER overcrowding risk). Let $d \in \mathbb{R}$ index calendar days and let the two-day window be

$$X = \{d, d+1\} \in \mathscr{P}_2^*(\mathbb{R}).$$

Interpret the membership value as the plausibility (normalized to $[0,1]$) that the *median ED wait time* stays within a safety threshold over X. Set a weekly baseline

$$r(d) = \frac{1 + \sin\left(\frac{2\pi d}{7}\right)}{2} \in [0,1],$$

and for three information sources $i \in \{1,2,3\}$ (historical EHR, simulation, and staffing), choose shifts and half-widths

$$(s_i, w_{i,1}, w_{i,2}) \in \{(0, 0.05, 0.08),\ (0.05, 0.08, 0.12),\ (0.10, 0.10, 0.15)\}.$$

Define two daily subintervals (morning j=1, evening j=2) by

$$[\underline{x}_{i,1}(X), \overline{x}_{i,1}(X)] = [\max\{0, r(d) - w_{i,1}\},\ \min\{1, r(d) + w_{i,1}\}],$$
$$[\underline{x}_{i,2}(X), \overline{x}_{i,2}(X)] = [\max\{0, r(d) + s_i - w_{i,2}\},\ \min\{1, r(d) + s_i + w_{i,2}\}].$$

Then, $\mathfrak{X}^{(2,2)}(X) = \{[\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] \mid i = 1,2,3;\ j = 1,2\}$ is a valid $(2,2)$-SuperHyperCloudy Number: *coverage* holds because $r(d)$ attains every $\alpha \in [0,1]$ as d varies (so $\alpha \in [\underline{x}_{i,1}(X), \overline{x}_{i,1}(X)]$ for some i and X), and *normality* holds because $\inf_{X,i,j} \underline{x}_{i,j}(X) = 0$ (take $r(d) = 0$) and $\sup_{X,i,j} \overline{x}_{i,j}(X) = 1$ (take $r(d) = 1$). For instance, at $d = 0$, we have $r(0) = \frac{1+\sin 0}{2} = \frac{1}{2}$, giving, e.g., for $i = 2$ (morning), the concrete interval $[0.50 - 0.08,\ 0.50 + 0.08] = [0.42,\ 0.58]$.

Example 19 ((1,2)-SuperHyperCloudy Number: flight on-time departure). Let $t \in [0,24]$ be the hour of day and take $X = \{t\} \in \mathscr{P}_1^*(\mathbb{R})$. Interpret the membership as the plausibility that a flight departs on time at hour t. Define a diurnal baseline

$$b(t) = \frac{1+\cos\left(\frac{\pi(t-7)}{12}\right)}{2} \in [0,1],$$

and three sources $i \in \{1,2,3\}$ (airport logs, airline ops, and weather). For each source, give two nested bands (baseline $j=1$ and disruption-aware $j=2$) with half-widths

$$(w_{i,1}, w_{i,2}) \in \{(0.05, 0.10),\ (0.10, 0.15),\ (0.15, 0.20)\}.$$

Set

$$[\underline{x}_{i,1}(X), \overline{x}_{i,1}(X)] = \left[\max\{0,\, b(t)-w_{i,1}\},\ \min\{1,\, b(t)+w_{i,1}\}\right],$$
$$[\underline{x}_{i,2}(X), \overline{x}_{i,2}(X)] = \left[\max\{0,\, b(t)-w_{i,2}\},\ \min\{1,\, b(t)+w_{i,2}\}\right].$$

Then, $\mathfrak{X}^{(1,2)}(\{t\}) = \{[\underline{x}_{i,j}(\{t\}), \overline{x}_{i,j}(\{t\})] \mid i = 1,2,3;\ j = 1,2\}$ is a valid $(1,2)$-SuperHyperCloudy Number: Because $b(t)$ ranges over $[0,1]$ as t varies, the union of all intervals covers $(0,1)$; moreover, $\inf \underline{x}_{i,j} = 0$ (e.g., $b(t) = 0$ with $j = 2$) and $\sup \overline{x}_{i,j} = 1$ (e.g., $b(t) = 1$ with $j = 2$), establishing normality. As a concrete numeric instance, at $t = 7$, we have $b(7) = \frac{1+\cos 0}{2} = 1$, so for $i = 1$ (baseline) $[\underline{x}_{1,1}, \overline{x}_{1,1}] = [1-0.05,\ 1] = [0.95,\ 1]$.

Theorem 6 *An (m,n)-SuperHyperCloudy Number generalizes an (m,n)-SuperHyperFuzzy Number and an (m,n)-SuperHyperCloudy Number reduces to a HyperCloudy Number when $m = n = 0$.*

Proof: By construction, setting $m = n = 0$ yields $\mathscr{P}_0^*(\mathbb{R}) = \mathbb{R}$ and collapses the interval family to a single layer, recovering exactly a HyperCloudy Number. Likewise, when each interval at the deepest level degenerates to a point, one recovers an (m,n)-SuperHyperFuzzy Number by the same argument as in the theorem.

Notation 1 Throughout, let $\mathfrak{X}^{(m,n)}$ be a fixed (m,n)-SuperHyperCloudy Number as in the definition, and for each $X \in \mathscr{P}_m^*(\mathbb{R})$, write

$$\mathfrak{X}^{(m,n)}(X) = \left\{ [\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] \ \middle|\ i \in I_X,\ j \in J_{X,i} \right\}.$$

Define the *two-layer envelope* at X by

$$\underline{X}(X) := \inf_{i \in I_X} \inf_{j \in J_{X,i}} \underline{x}_{i,j}(X), \qquad \overline{X}(X) := \sup_{i \in I_X} \sup_{j \in J_{X,i}} \overline{x}_{i,j}(X). \quad (1)$$

Theorem 7 (Envelope on $\mathscr{P}_m^*(\mathbb{R})$ is a cloud). *The map*

$$X \longmapsto [\underline{X}(X), \overline{X}(X)] \qquad (X \in \mathscr{P}_m^*(\mathbb{R}))$$

satisfies coverage and normality on the range $[0,1]$, i.e.,

$$(0,1) \subseteq \bigcup_{X \in \mathscr{P}_m^*(\mathbb{R})} [\underline{X}(X), \overline{X}(X)] \subseteq [0,1],$$

$$\sup_X \overline{X}(X) = 1, \ \inf_X \underline{X}(X) = 0.$$

Proof: Fix X. For any component interval $[\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)]$, we have, by Equation (1),

$$\underline{X}(X) = \inf_{i',j'} \underline{x}_{i',j'}(X) \le \underline{x}_{i,j}(X), \qquad \overline{x}_{i,j}(X) \le \sup_{i',j'} \overline{x}_{i',j'}(X) = \overline{X}(X),$$

hence $[\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] \subseteq [\underline{X}(X), \overline{X}(X)]$. Taking the union over all X, i, j yields

$$\bigcup_{X,i,j} [\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] \subseteq \bigcup_X [\underline{X}(X), \overline{X}(X)].$$

By coverage of $\mathfrak{X}^{(m,n)}$,

$$(0,1) \subseteq \bigcup_{X,i,j} [\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] \subseteq [0,1],$$

so the same inclusion holds for the envelope union. Normality follows from

$$\sup_X \overline{X}(X) = \sup_X \sup_{i,j} \overline{x}_{i,j}(X) = 1, \qquad \inf_X \underline{X}(X) = \inf_X \inf_{i,j} \underline{x}_{i,j}(X) = 0,$$

which are exactly the normality conditions of $\mathfrak{X}^{(m,n)}$.

Theorem 8 (Minimal dominating property). *Let $Y(X) = [\ell(X), u(X)]$ be any interval assignment on $\mathscr{P}_m^*(\mathbb{R})$ such that for every X and every component*

$$[\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] \subseteq [\ell(X), u(X)].$$

Then, for all X,

$$[\underline{X}(X), \overline{X}(X)] \subseteq [\ell(X), u(X)].$$

In particular, the envelope $[\underline{X}, \overline{X}]$ is the smallest (pointwise) such dominating assignment.

Proof: From the hypothesis,

$$\ell(X) \le \underline{x}_{i,j}(X) \quad \forall i,j \Rightarrow \ell(X) \le \inf_{i,j} \underline{x}_{i,j}(X) = \underline{X}(X),$$

and similarly

$$\overline{x}_{i,j}(X) \le u(X) \quad \forall i,j \Rightarrow \sup_{i,j} \overline{x}_{i,j}(X) = \overline{X}(X) \le u(X).$$

Hence, the inclusion of intervals holds for each X.

Theorem 9 (α-cut characterization on $\mathscr{P}_m^*(\mathbb{R})$). *For $\alpha \in (0,1]$, define*

$$\mathscr{T}_\alpha := \left\{ X \in \mathscr{P}_m^*(\mathbb{R}) \ \middle|\ \exists i \in I_X,\ j \in J_{X,i} : \underline{x}_{i,j}(X) \le \alpha \le \overline{x}_{i,j}(X) \right\}.$$

Then

$$\mathscr{T}_\alpha = \left\{ X \in \mathscr{P}_m^*(\mathbb{R}) \ \middle|\ \underline{X}(X) \le \alpha \le \overline{X}(X) \right\}.$$

Proof: "$\subseteq$": If $X \in \mathscr{T}_\alpha$, pick (i,j) with $\underline{x}_{i,j}(X) \le \alpha \le \overline{x}_{i,j}(X)$. Then

$$\underline{X}(X) = \inf_{i',j'} \underline{x}_{i',j'}(X) \le \underline{x}_{i,j}(X) \le \alpha,$$

$$\alpha \le \overline{x}_{i,j}(X) \le \sup_{i',j'} \overline{x}_{i',j'}(X) = \overline{X}(X),$$

so X satisfies the envelope inequality.

"$\supseteq$": If $\underline{X}(X) \le \alpha \le \overline{X}(X)$, by the definitions of infimum/supremum, we can choose sequences (i_n, j_n) and (i'_n, j'_n) such that

$$\underline{x}_{i_n,j_n}(X) \downarrow \underline{X}(X) \quad \text{and} \quad \overline{x}_{i'_n,j'_n}(X) \uparrow \overline{X}(X).$$

Fix n large enough that

$$\underline{x}_{i_n,j_n}(X) \le \underline{X}(X) + \tfrac{1}{n} \le \alpha \quad \text{and} \quad \alpha \le \overline{X}(X) + \tfrac{1}{n} \ge \overline{x}_{i'_n,j'_n}(X).$$

Let $(\hat{i}, \hat{j})$ be either (i_n, j_n) or (i'_n, j'_n) for which both inequalities hold (pick the one giving the tighter lower and upper conditions; this is always possible after enlarging n if necessary). Then

$$\underline{x}_{\hat{i},\hat{j}}(X) \le \alpha \le \overline{x}_{\hat{i},\hat{j}}(X),$$

so $X \in \mathscr{T}_\alpha$.

Theorem 10 (Union closure and envelope formula). *Let $\mathfrak{X}^{(m,n)}$ and $\mathfrak{Y}^{(m,n)}$ be (m,n)-SuperHyperCloudy Numbers. Define their pointwise union*

$$(\mathfrak{X} \sqcup \mathfrak{Y})(X) \;:=\; \mathfrak{X}^{(m,n)}(X) \,\cup\, \mathfrak{Y}^{(m,n)}(X).$$

Then, $\mathfrak{X} \sqcup \mathfrak{Y}$ is an (m,n)-SuperHyperCloudy Number, and its envelope satisfies

$$\underline{X \sqcup Y}(X) \;=\; \min\{\underline{X}(X),\, \underline{Y}(X)\}, \qquad \overline{X \sqcup Y}(X) \;=\; \max\{\overline{X}(X),\, \overline{Y}(X)\}.$$

Proof: Nonemptiness at each X is clear as each family is nonempty. Coverage:

$$\bigcup_X \bigcup((\mathfrak{X} \sqcup \mathfrak{Y})(X)) \;=\; \Big(\bigcup_X \bigcup \mathfrak{X}(X)\Big) \cup \Big(\bigcup_X \bigcup \mathfrak{Y}(X)\Big) \;\supseteq\; (0,1).$$

Normality holds because

$$\sup_X \overline{X \sqcup Y}(X) = \sup_X \max\{\overline{X}(X), \overline{Y}(X)\}$$

$$= \max\{\sup_X \overline{X}(X), \sup_X \overline{Y}(X)\} = \max\{1,1\} = 1,$$

$$\inf_X \underline{X \sqcup Y}(X) = \inf_X \min\{\underline{X}(X), \underline{Y}(X)\}$$

$$\le \min\{\inf_X \underline{X}(X), \inf_X \underline{Y}(X)\} = \min\{0,0\} = 0,$$

and trivially $\underline{X \sqcup Y}(X) \ge 0$. The envelope formulas follow from taking $\inf/\sup$ over the union of sets of endpoints:

$$\inf(A \cup B) = \min\{\inf A, \inf B\}, \quad \sup(A \cup B) = \max\{\sup A, \sup B\}.$$

Theorem 11 (Level-n flattening to $(m,1)$). *Define the n-*flattening $\mathrm{Flat}_n(\mathfrak{X}^{(m,n)})$ *by*

$$\mathrm{Flat}_n(\mathfrak{X})(X) \;:=\; \Big\{ \big[\inf_{j \in J_{X,i}} \underline{x}_{i,j}(X),\; \sup_{j \in J_{X,i}} \overline{x}_{i,j}(X) \big] \;\Big|\; i \in I_X \Big\}.$$

Then, $\mathrm{Flat}_n(\mathfrak{X})$ *is an $(m,1)$-SuperHyperCloudy Number, and its envelope coincides with Equation (1):*

$$\inf_i \inf_j \underline{x}_{i,j}(X) \;=\; \inf_i \inf_j \underline{x}_{i,j}(X) \;=\; \inf_i \Big(\inf_j \underline{x}_{i,j}(X) \Big),$$

$$\sup_i \sup_j \overline{x}_{i,j}(X) \;=\; \sup_i \sup_j \overline{x}_{i,j}(X) \;=\; \sup_i \Big(\sup_j \overline{x}_{i,j}(X) \Big).$$

Consequently, coverage and normality are preserved under Flat_n.

Proof: Nonemptiness: for each $i \in I_X$, the inner $\inf/\sup$ are taken over a nonempty $J_{X,i}$, producing one interval; thus, the set over i is nonempty. Each produced interval lies in $[0,1]$ and is closed, as $\inf \le \sup$ and endpoints lie in $[0,1]$. The envelope equalities are identities of $\inf/\sup$ over a finite or arbitrary two-level index set. Coverage/normality follow Theorem 7 because the envelope did not change.

Theorem 12 (Singleton embedding yields a HyperCloudy Number). *Let $\iota_m : \mathbb{R} \to \mathscr{P}^*_m(\mathbb{R})$ denote the m-fold singleton embedding*

$$\iota_1(t) = \{t\}, \quad \iota_{k+1}(t) = \{\iota_k(t)\}.$$

Then, $H(t) := \mathrm{Flat}_n(\mathfrak{X}^{(m,n)})(\iota_m(t))$ defines a HyperCloudy Number on $\mathbb{R}$.

Proof: For each t, $X = \iota_m(t)$ belongs to $\mathscr{P}^*_m(\mathbb{R})$; hence, $\mathrm{Flat}_n(\mathfrak{X})(X)$ is a nonempty family of closed subintervals of $[0,1]$. The global coverage and normality for H follow those of $\mathrm{Flat}_n(\mathfrak{X})$ and the fact that the image of ι_m is nonempty and (as t varies) covers all embedded singletons, so

$$(0,1) \;\subseteq\; \bigcup_X \bigcup \mathrm{Flat}_n(\mathfrak{X})(X) \;\subseteq\; [0,1]$$

$$\Rightarrow\; (0,1) \;\subseteq\; \bigcup_t \bigcup H(t) \;\subseteq\; [0,1].$$

Thus, $t \mapsto H(t)$ is a HyperCloudy Number.

Theorem 13 (Monotone pushforward). *Let $g\colon [0,1] \to [0,1]$ be nondecreasing with $g(0) = 0$, $g(1) = 1$. Define*

$$(g_\# \mathfrak{X})(X) \;:=\; \Big\{ [\,g(\underline{x}_{i,j}(X)),\, g(\overline{x}_{i,j}(X))\,] \;\Big|\; i \in I_X,\; j \in J_{X,i} \Big\}.$$

Then, $g_\# \mathfrak{X}$ is an (m,n)-SuperHyperCloudy Number with envelope

$$\underline{(g_\# X)}(X) \;=\; g\big(\underline{X}(X)\big), \qquad \overline{(g_\# X)}(X) \;=\; g\big(\overline{X}(X)\big).$$

Proof: g preserves order and closed intervals: $g([\ell,u]) = [g(\ell), g(u)] \subseteq [0,1]$. Coverage:

$$\bigcup_X \bigcup_{i,j} [g(\underline{x}_{i,j}(X)), g(\overline{x}_{i,j}(X))]$$

$$= g\Big(\bigcup_X \bigcup_{i,j} [\underline{x}_{i,j}(X), \overline{x}_{i,j}(X)] \Big) \;\supseteq\; g((0,1)) = (0,1).$$

Normality:

$$\sup_X \overline{(g_\# X)}(X) = \sup_X \sup_{i,j} g(\overline{x}_{i,j}(X))$$

$$= g\Big(\sup_X \sup_{i,j} \overline{x}_{i,j}(X) \Big) = g(1) = 1,$$

$$\inf_X \underline{(g_\# X)}(X) = \inf_X \inf_{i,j} g(\underline{x}_{i,j}(X)) = g\Big(\inf_X \inf_{i,j} \underline{x}_{i,j}(X) \Big) = g(0) = 0,$$

using monotonicity of g and properties of $\inf/\sup$. The envelope identities follow directly:

$$\underline{(g_\# X)}(X) = \inf_{i,j} g(\underline{x}_{i,j}(X)) = g\Big(\inf_{i,j} \underline{x}_{i,j}(X) \Big) = g(\underline{X}(X)),$$

$$\overline{(g_\# X)}(X) = \sup_{i,j} g(\overline{x}_{i,j}(X)) = g\Big(\sup_{i,j} \overline{x}_{i,j}(X) \Big) = g(\overline{X}(X)).$$

4. Additional Results: Ordered HyperFuzzy, SuperHyperFuzzy, and Cloudy Numbers

In this section, as further results of this paper, we extend the concept of the Ordered Fuzzy Number by introducing the Ordered HyperFuzzy Number, the Ordered SuperHyperFuzzy Number, and the Ordered Cloudy Number. An *Ordered HyperFuzzy Number* assigns each value a set of plausible memberships, which remain monotone under a specified order, thereby aggregating multisource uncertainty. An *Ordered SuperHyperFuzzy Number* maps higher-level subsets to families of membership sets while preserving monotonicity under order, representing nested and hierarchical uncertainty. An *Ordered Cloudy Number* assigns each value a membership interval whose endpoints vary monotonically under order, encoding lower–upper bounds that guide decision-making processes.

Definition 15 (Ordered Fuzzy Number) [73–76]. An *Ordered Fuzzy Number* (also called an *ordered fuzzy real*) is an ordered pair

$$A = (\mu_A^{\uparrow}, \mu_A^{\downarrow}),$$

where the functions $\mu_A^{\uparrow}, \mu_A^{\downarrow} \colon [0,1] \to \mathbb{R}$ are continuous. The corresponding membership curves are

$$\mathrm{up}_A = \{(\mu_A^{\uparrow}(y), y) : y \in [0,1]\}, \qquad \mathrm{down}_A = \{(\mu_A^{\downarrow}(y), y) : y \in [0,1]\}.$$

Notation 2 Let $\mathscr{I} = \{[a,b] \subseteq \mathbb{R} \mid a \le b\}$ be a set of all nonempty closed intervals in $\mathbb{R}$. For $I = [a,b] \in \mathscr{I}$, write $\underline{I} = a$ and $\overline{I} = b$.

Definition 16 (Ordered HyperFuzzy Number). An Ordered HyperFuzzy Number is a map

$$\mathscr{H} : [0,1] \longrightarrow \mathscr{P}(\mathscr{I}) \setminus \{\emptyset\}, \qquad \alpha \longmapsto \mathscr{H}(\alpha),$$

satisfying:

1) **Closed-interval values:** $\mathscr{H}(\alpha) \subseteq \mathscr{I}$ and $\mathscr{H}(\alpha) \neq \emptyset$ for all α.

2) **Ordered envelope (nested α-cuts):** If

$$L_{\mathscr{H}}(\alpha) := \inf\{\underline{I} \mid I \in \mathscr{H}(\alpha)\},$$

$$U_{\mathscr{H}}(\alpha) := \sup\{\overline{I} \mid I \in \mathscr{H}(\alpha)\},$$

then for $0 \le \alpha \le \beta \le 1$,

$$L_{\mathscr{H}}(\alpha) \le L_{\mathscr{H}}(\beta),$$

$$U_{\mathscr{H}}(\alpha) \ge U_{\mathscr{H}}(\beta),$$

and $L_{\mathscr{H}}(\alpha) \le U_{\mathscr{H}}(\alpha)$ for all α.

The *envelope α-cut* is the interval $\mathscr{H}_\alpha^{\mathrm{env}} = [L_{\mathscr{H}}(\alpha), U_{\mathscr{H}}(\alpha)]$.

Example 20 (Ordered HyperFuzzy Number: on-time commute preference). Let the universe be travel time $T = [0,90]$ (minutes). Define a preference order

$$s \preceq_T t \iff s \le t$$

("earlier is no worse"). For three independent predictors, set

$$\mu_1(t) = \max\{0,\, 1 - \tfrac{t}{60}\},$$

$$\mu_2(t) = \max\{0,\, 1 - \tfrac{t}{50}\},$$

$$\mu_3(t) = \max\{0,\, 1 - \tfrac{t}{40}\}.$$

The Ordered HyperFuzzy Number $\widetilde{A}$ is

$$\widetilde{\mu}(t) \;=\; \{\mu_1(t), \mu_2(t), \mu_3(t)\} \subset [0,1] \qquad (t \in T).$$

Order consistency. If $s \preceq_T t$ (i.e. $s \le t$), then for each k,

$$1 - \tfrac{s}{b_k} \;\ge\; 1 - \tfrac{t}{b_k} \;\Rightarrow$$

$$\mu_k(s) \;=\; \max\{0, 1 - \tfrac{s}{b_k}\} \;\ge\; \max\{0, 1 - \tfrac{t}{b_k}\} \;=\; \mu_k(t),$$

so the set of plausible memberships is monotone w.r.t. $\preceq_T$. Hence, $\widetilde{A}$ is an Ordered HyperFuzzy Number.

Theorem 14 (OHFN generalizes OFN). *Let $A = (L_A, U_A)$ be an Ordered Fuzzy Number. Define $\mathscr{H}(\alpha) := \{[L_A(\alpha), U_A(\alpha)]\}$ for each $\alpha \in [0,1]$. Then, $\mathscr{H}$ is an Ordered HyperFuzzy Number and its envelope satisfies*

$$L_{\mathscr{H}}(\alpha) = L_A(\alpha), \qquad U_{\mathscr{H}}(\alpha) = U_A(\alpha),$$

so $\mathscr{H}_\alpha^{\mathrm{env}} = A_\alpha$ for all α.

Proof: By construction, $\mathscr{H}(\alpha)$ is a nonempty set of closed intervals, so 1) holds. For 2),

$$L_{\mathscr{H}}(\alpha) = \inf\{\underline{I} : I \in \mathscr{H}(\alpha)\} = \inf\{L_A(\alpha)\} = L_A(\alpha),$$

$$U_{\mathscr{H}}(\alpha) = \sup\{\overline{I} : I \in \mathscr{H}(\alpha)\} = \sup\{U_A(\alpha)\} = U_A(\alpha).$$

Monotonicity and $L_A(\alpha) \le U_A(\alpha)$ therefore transfer to $L_{\mathscr{H}}, U_{\mathscr{H}}$, proving 3). Hence, $\mathscr{H}_\alpha^{\mathrm{env}} = [L_A(\alpha), U_A(\alpha)] = A_\alpha$.

Definition 17 (Ordered SuperHyperFuzzy Number). An Ordered SuperHyperFuzzy Number is a map

$$\mathscr{S} : [0,1] \longrightarrow \mathscr{P}\big(\mathscr{P}(\mathscr{I}) \setminus \{\emptyset\}\big) \setminus \{\emptyset\}, \qquad \alpha \longmapsto \mathscr{S}(\alpha),$$

such that with the two-layer envelope

$$L_{\mathscr{S}}(\alpha) := \inf\Big\{\underline{I} \;\Big|\; \exists \mathscr{F} \in \mathscr{S}(\alpha) \text{ and } I \in \mathscr{F}\Big\},$$

$$U_{\mathscr{S}}(\alpha) := \sup\Big\{\overline{I} \;\Big|\; \exists \mathscr{F} \in \mathscr{S}(\alpha) \text{ and } I \in \mathscr{F}\Big\},$$

we have, for $0 \le \alpha \le \beta \le 1$,

$$L_{\mathscr{S}}(\alpha) \le L_{\mathscr{S}}(\beta), \qquad U_{\mathscr{S}}(\alpha) \ge U_{\mathscr{S}}(\beta), \qquad L_{\mathscr{S}}(\alpha) \le U_{\mathscr{S}}(\alpha).$$

The *envelope α-cut* is $\mathscr{S}_\alpha^{\mathrm{env}} = [L_{\mathscr{S}}(\alpha), U_{\mathscr{S}}(\alpha)]$.

Example 21 (Ordered SuperHyperFuzzy Number: two-task deadline reliability). Let the elements of $\mathscr{P}_2^*(\mathbb{R})$ be two-point sets $X = \{t_1, t_2\}$ with $t_1 \le t_2$ (task completion times). Define the order

$$X \preceq Y \iff \max X \le \max Y \quad \text{(earlier worst-case finish is no worse).}$$

For three evaluators $i \in \{1,2,3\}$ and two modes $j \in \{1,2\}$ (optimistic/conservative), put

$$m_{i,1}(X) = \max\{0,\, 1 - \tfrac{\max X}{B_{i,1}}\}, \qquad m_{i,2}(X) = \max\{0,\, 1 - \tfrac{\max X}{B_{i,2}}\},$$

with, e.g., $(B_{1,1}, B_{1,2}) = (60,45)$, $(B_{2,1}, B_{2,2}) = (55,40)$, $(B_{3,1}, B_{3,2}) = (50,35)$. Define the $(2,2)$-SuperHyperFuzzy mapping by

$$\widetilde{\mu}_{2,2}(X) \;=\; \big\{\, \{m_{i,1}(X), m_{i,2}(X)\} \;:\; i = 1,2,3 \,\big\} \;\subset \mathscr{P}_2([0,1]).$$

Order consistency. If $X \preceq Y$, then $\max X \le \max Y$; hence, for all (i,j),

$$1 - \tfrac{\max X}{B_{i,j}} \;\ge\; 1 - \tfrac{\max Y}{B_{i,j}} \;\Rightarrow\; m_{i,j}(X) \;\ge\; m_{i,j}(Y).$$

Thus, each inner set and the whole family are monotone in the order $\preceq$, so $\widetilde{\mu}_{2,2}$ is an *Ordered* SuperHyperFuzzy Number. (For example, if $\max X = 20$ and $\max Y = 30$ with $B_{1,1} = 60$, then $m_{1,1}(X) = 1 - 20/60 = 0.666\ldots \ge 0.5 = 1 - 30/60 = m_{1,1}(Y)$.)

Theorem 15 (OSHFN generalizes OFN). *Let $A = (L_A, U_A)$ be an Ordered Fuzzy Number. Define*

$$\mathscr{S}(\alpha) := \{ \{[L_A(\alpha), U_A(\alpha)]\} \} \qquad (\alpha \in [0,1]).$$

Then, $\mathscr{S}$ is an Ordered SuperHyperFuzzy Number and

$$L_{\mathscr{S}}(\alpha) = L_A(\alpha), \qquad U_{\mathscr{S}}(\alpha) = U_A(\alpha),$$

hence $\mathscr{S}^{\mathrm{env}}_{\alpha} = A_\alpha$.

Proof: For each α, $\mathscr{S}(\alpha)$ is a nonempty set whose sole element is the nonempty family $\{[L_A(\alpha), U_A(\alpha)]\}$; thus, the value of $\mathscr{S}$ lies in $\mathscr{P}(\mathscr{P}(\mathscr{I}) \setminus \{\emptyset\}) \setminus \{\emptyset\}$. By the same calculation as in the proof of Theorem 14,

$$L_{\mathscr{S}}(\alpha) = \inf\{L_A(\alpha)\} = L_A(\alpha), \qquad U_{\mathscr{S}}(\alpha) = \sup\{U_A(\alpha)\} = U_A(\alpha),$$

and the nesting and order properties follow.

Definition 18 (Ordered Cloudy Number). An Ordered Cloudy Number is a pair of interval-valued maps

$$\mathscr{C} = (\mathscr{C}^-, \mathscr{C}^+),$$

$$\mathscr{C}^{\pm} : (0,1] \to \mathscr{I},$$

such that for all $0 < \alpha \leq \beta \leq 1$:

1) **Nestedness (each branch):** $\mathscr{C}^{\pm}(\beta) \subseteq \mathscr{C}^{\pm}(\alpha)$. Writing $\mathscr{C}^{\pm}(\alpha) = [L^{\pm}(\alpha), U^{\pm}(\alpha)]$, this is equivalent to

$$L^{\pm}(\alpha) \leq L^{\pm}(\beta) \quad \text{and}$$

$$U^{\pm}(\alpha) \geq U^{\pm}(\beta).$$

2) **Lower vs. upper inclusion:**

$$\mathscr{C}^-(\alpha) \subseteq \mathscr{C}^+(\alpha)$$

(i.e., $L^+(\alpha) \leq L^-(\alpha) \leq U^-(\alpha) \leq U^+(\alpha)$).

Intuitively, $\mathscr{C}^-(\alpha)$ and $\mathscr{C}^+(\alpha)$ may be viewed as *necessity* and *possibility* α-cuts induced by lower/upper membership bounds of an underlying cloud.

Example 22 (Ordered Cloudy Number: price acceptability). Let price $p \in [0,\infty)$ with order $p \preceq q \iff p \leq q$ (a lower price is no worse). Define the Cloudy Number x by

$$x(p) = [\underline{x}(p), \overline{x}(p)],$$

$$\underline{x}(p) = \max\{0,\ 1 - \tfrac{p}{200}\},$$

$$\overline{x}(p) = \max\{0,\ 1 - \tfrac{p}{100}\}.$$

Order consistency. If $p \preceq q$, then $p \leq q$, so

$$1 - \tfrac{p}{200} \ \geq\ 1 - \tfrac{q}{200} \ \Rightarrow\ \underline{x}(p) \geq \underline{x}(q),$$

$$1 - \tfrac{p}{100} \ \geq\ 1 - \tfrac{q}{100} \ \Rightarrow\ \overline{x}(p) \geq \overline{x}(q).$$

Hence, $x(p)$ shrinks monotonically as price increases, making x an Ordered Cloudy Number that encodes price acceptability under the stated preference.

Theorem 16 (OCN generalizes OFN). *Let $A = (L_A, U_A)$ be an Ordered Fuzzy Number. Define an Ordered Cloudy Number by*

$$\mathscr{C}^-(\alpha) = \mathscr{C}^+(\alpha) = [L_A(\alpha), U_A(\alpha)] \qquad (\alpha \in (0,1]).$$

Then, $\mathscr{C}$ is an Ordered Cloudy Number, and A is recovered by

$$L_A(\alpha) = \inf \mathscr{C}^{\pm}(\alpha), \qquad U_A(\alpha) = \sup \mathscr{C}^{\pm}(\alpha).$$

Proof: L_A is nondecreasing and U_A is nonincreasing. Hence, for $0 < \alpha \leq \beta \leq 1$,

$$[L_A(\beta), U_A(\beta)] \subseteq [L_A(\alpha), U_A(\alpha)],$$

so nestedness 1) holds for both branches. Because the two branches coincide by construction, $\mathscr{C}^-(\alpha) = \mathscr{C}^+(\alpha)$, inclusion 2) is trivially true. Finally,

$$\inf \mathscr{C}^{\pm}(\alpha) = L_A(\alpha), \qquad \sup \mathscr{C}^{\pm}(\alpha) = U_A(\alpha),$$

so the Ordered Fuzzy Number A is recovered exactly from $\mathscr{C}$.

5. Conclusion

In this paper, we introduced two new constructs: the *HyperCloudy Number* and the (m,n)*-SuperHyperCloudy Number*. In future work, we are planning to explore extensions leveraging Neutrosophic Sets [13, 77, 78], Bipolar Fuzzy Sets [79], Three-way Fuzzy Sets [80, 81], m-polar Fuzzy Sets [82], q-Rung Orthopair Fuzzy Sets [83], Plithogenic Sets [84, 85], Hesitant Fuzzy Sets [86, 87], Spherical Fuzzy Sets [88, 89], Z-Numbers [90, 91], and related frameworks. In addition, we aim to investigate possible extensions of the concepts introduced in this paper by employing graphs [92], hypergraphs [93–96], and superhypergraphs [97–100]. We also anticipate further research on practical applications, the development of programming tools, computational experiments, and algorithmic improvements.

For example, future research could explore applications of Cloudy Numbers in decision theory [101–103] and risk modeling [104, 105].

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Ethical Statement

This study does not contain any studies with human or animal subjects performed by the author.

Conflicts of Interest

The author declares that he has no conflicts of interest to this work.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study. This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Author Contribution Statement

Takaaki Fujita: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing – original draft, Writing – review & editing, Visualization, Supervision, Project administration.

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How to Cite: Fujita, T. (2026). HyperCloudy Number and (m,n)-Super-HyperCloudy Number. *Journal of Data Science and Intelligent Systems*. https://doi.org/10.47852/bonviewJDSIS62027651