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Efficiency Analysis and Ranking of the Decision-Making Units Using Bipolar Fuzzy Data Envelopment Analysis

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Abstract: This manuscript introduces a novel extension of Fuzzy Data Envelopment Analysis (FDEA) by incorporating Triangular Bipolar Fuzzy Numbers (BTFNs), thereby proposing the Bipolar Fuzzy DEA (BFDEA) framework to better model environments with uncertainty and conflicting information. The authors define a magnitude function to transform the proposed BFDEA model into a crisp linear programming model, enabling the computation of efficiency scores for Decision-Making Units (DMUs). To address the challenge of ranking fully efficient units, a bipolar super-efficiency model is developed, complemented by a benchmarking methodology that identifies peer DMUs to guide performance improvement of each inefficient DMU. The approach is clearly illustrated through a numerical example involving ten DMUs, with detailed evaluation, ranking, and benchmarking results and compared with the existing fuzzy model. The paper contributes significantly to the literature by enhancing DEA's ability to handle both positive and negative aspects of fuzziness, thus offering a more expressive and realistic efficiency analysis framework in complex decision-making environments.

Keywords: bipolar fuzzy set, data envelopment analysis, triangular bipolar fuzzy number, super efficiency model, benchmarking

1. Introduction

Fuzzy set theory, introduced by Zadeh [1], expands upon classical set theory by allowing elements to possess degrees of membership, instead of being classified only in a binary way. This new framework gives an effective means for analyzing the uncertainty and imprecision that naturally occurs in many real-world situations, making it an important part of the decision-making process in artificial intelligence, control systems, and operations research [2, 3]. A bipolar fuzzy set (BFS) [4], which expands upon the principles of fuzzy set theory, serves as a powerful tool for capturing and representing uncertainty and ambiguity in a more nuanced manner. By associating each element with degrees of positive membership and negative membership, this framework is able to accurately reflect the inherent bipolarity that is often encountered in real-world situations. The significance of this dual representation becomes evident when considering its applications in decision-making processes and expert systems, where the ability to effectively handle imprecise and conflicting information is crucial [5]. In particular, the utilization of BFSs offers a more realistic depiction of uncertainty in various decision-making scenarios, as it takes into account both positive and negative membership degrees. This enhanced representation greatly enhances the expressiveness of models in fields such as expert systems, decision analysis, and artificial intelligence, where a comprehensive understanding of uncertainty is vital in order to facilitate effective decision-making [6]. The utilization of BFSs in these domains not only allows for a more accurate representation of uncertainty, but also contributes to the overall effectiveness of decision-making processes by providing a comprehensive understanding of the complexities involved. Also, in the context of group decision-making, where diverse opinions may result in conflicting preferences, BFSs provide a flexible framework to model the varying degrees of agreement and disagreement among decision-makers [7]. This enhances the adaptability and effectiveness of decision support systems in handling the intricacies of real-world complexities, ultimately improving the overall robustness of decision outcomes. The extensive applications and benefits of BFSs make them an invaluable tool in addressing uncertain problems across a range of industries and domains, empowering decision-makers to make more informed and effective choices [6, 8, 9].

Data Envelopment Analysis (DEA) provides a solution to evaluate the performance of complex processes through a framework that evaluates the relative efficiency of comparable decision-making units (DMUs) that have multiple input–output variables. First introduced by Charnes et al. [10], traditional or classical DEA has established itself as a solid foundation in the fields of operations research and management science due to its non-parametric approach in efficiency measurement known as CCR model. This was followed to introduce the BCC model by Banker et al. [11], which added the concepts of variable returns to scale to the original CCR model. Following that, a number of DEA models are created these are Additive, SBM, Super efficiency, Network DEA, Dynamic DEA, etc. [12, 13]. A novel SET-DEA-IDEA model for objectively evaluating and ranking equities is developed and applied to Taiwan's tourism sector [14]. It identifies efficient stocks and ranks them by adaptability, offering a robust tool for financial decision-making. These different models validated DEA as an important method of assessing the relative efficiency of DMUs. Even though traditional DEA did not solve all efficiency measurement problems, its limitation of handling imprecise and uncertain data has led to fuzzy DEA models that use fuzzy set theory to capture vagueness in applications [15, 16]. The notion of fuzzy sets, first proposed by Zadeh [1], established a theoretical basis for these advancements. Fuzzy DEA [17] has become an essential instrument in several domains, such as finance and healthcare, providing a method to evaluate efficiency in situations characterized by imprecise or ambiguous data [18, 19]. As research

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on fuzzy DEA advanced, researchers started to investigate extended fuzzy DEA models, which integrate extra complexity and flexibility in efficiency analysis. The shift towards fuzzy DEA provides a crucial methodological step forward that allows researchers and practitioners to measure data-driven decisions that include inherent uncertainties. Fuzzy DEA models have become fundamental instruments across multiple areas of practice such as finance, health, and agriculture, resulting in better assessments of efficiency when data is uncertain or subjective [19–26]. Recent research has demonstrated the potential regulatory implications of fuzzy DEA in the development of organizational strategies and policies, suggesting its increasing importance in the space of operational efficiency measurement [27]. Table 1 summarizes several FDEA studies, which show that earlier research employs varied solution strategies for analyzing efficiencies, but it doesn't generally provide a complete ranking of units nor benchmarking recommendations. The proposed study introduces a BFDEA model and develops a magnitude value solution approach to fill this gap and overcome two important literature limitations.

The purpose of this study is to explore the motivation behind the analysis of efficiency measurement and complete ranking technique in a bipolar fuzzy environment. The reason for incorporating a BFS into the framework of DEA arises from the recognition of the limitations of traditional DEA models when faced with uncertainties and imprecise information. In many real-world scenarios, DMUs operate in environments that are characterized by ambiguity and vagueness, which poses a significant challenge in accurately assessing their efficiency using conventional approaches. The introduction of a BFS in DEA is driven by the need to capture both positive and negative aspects of uncertainty, acknowledging that DMUs may exhibit both excellence and inefficiencies. BFSs offer a more expressive and flexible representation of uncertainty compared to standard fuzzy sets. By allowing membership degrees to span across positive and negative values, they enable a nuanced modeling of imprecision, making them highly suitable for situations where decision-makers need to consider not only positive indicators of efficiency but also potential inefficiencies or deviations from the ideal performance. The incorporation of BFSs into DEA enhances the model's ability to handle conflicting and contradictory information, providing decision-makers with a more comprehensive and realistic assessment of DMUs' efficiency in uncertain environments. This approach is particularly valuable in sectors where decision-making involves inherent vagueness or when performance evaluations require considering both positive and negative deviations from the optimal efficiency frontier. Ultimately, the integration of BFSs into DEA contributes to the establishment of a robust and reliable framework for measuring and enhancing the performance of DMUs in complex and fuzzy settings.

The main contribution of this manuscript is emphasized in the subsequent manner.

1) The magnitude function for triangular bipolar fuzzy number is developed based on the left and right magnitude of TBFN.
2) The bipolar fuzzy DEA model proposed by considering the inputs and outputs are TBFNs.
3) The magnitude function is used to solve the proposed BFDEA model to evaluate the relative efficiency of the DMUs in an uncertain condition.
4) The bipolar fuzzy super-efficiency model is proposed to completely rank the DMUs when multiple DMUs are efficient.
5) The Reference Set for BFDEA model is proposed to find the benchmarking units for each DMUs.

The rest of the manuscript is organized as follows: Section 2 presents the literature review on the bipolar fuzzy set, its applications in various disciplines. Section 3 provides the preliminaries of bipolar sets and its arithmetic properties, definition on the magnitude function and properties. Section 4 discusses the proposed BFDEA model, complete ranking and benchmarking techniques. Section 5 discusses details of the solution procedure by presenting an algorithm and flowchart. Section 6 illustrates a numerical example, to demonstrate how to use the suggested BFDEA model for efficiency analysis, complete ranking and benchmarking. Finally in Section 7, we provide the conclusions while discussing the limitations and promising avenues for further investigation.

2. Literature Review

The growing complexity and uncertainty of decision-making settings have propelled the advancement of fuzzy set theory towards more articulate frameworks. One such advancement is the BFSs, which records both the positive and negative preferences of people who make decisions. Zhang [4] proposed the concept to more accurately represent human cognitive processes, wherein individuals frequently evaluate situations in terms of advantages and disadvantages. Since that time, researchers have used BFSs in many areas of engineering, mathematics, science, medical, and economics, especially in multi-criteria decision-making (MCDM). Zararsiz and Riaz [28] presented the idea of bipolar fuzzy metric spaces with continuous t-norms and bipolar continuous symmetry ts-conorms, and discussed some topological and functional results. In addition, it offered a MADM-based application for choosing the best COVID-19 vaccine through similarity measures of bipolar fuzzy metric spaces. Liu et al. [29] proposed Occupational Health and Safety (OHS) model by integrating BFSs and MABAC method to determine the risk ranking of hazards. Riaz and Tehrim [30] proposed a novel MAGDM approach which extended the VIKOR method on

Table 1
Comparison of recent studied on fuzzy DEA

Source	Data type	Model	Solution approach	Complete ranking	Bench-marking
[19]	TFN	Malquist DEA	Expected value	No	No
[20]	IFN	CCR & BCC	Minimax regret approach	No	No
[21]	TFN	CCR & Super efficiency	Score value	Yes	No
[22]	PFN	Dynamic DEA	α-cut approach	No	No
[23]	TFN	CCR	Multi objective	Yes	No
[24]	TFN	Network DEA	α-cut approach	No	No
[25]	STrFN	CCR	Parametric approach	No	No
[26]	TFN	Cross efficiency	α-cut approach	Yes	No
This work	TBFN	CCR & Super efficieny	Magnitude value	Yes	Yes

the basis of connection number (CN)-based metrics coined from bipolar fuzzy sets (BFS). It demonstrated novel metric spaces for CNs of BFNs and implemented these to improve decision-making under uncertainty, corroborating usefulness with examples and a comparative approach. Mustafa et al. [31] developed the new idea to assist students in identifying the right university by assessing the key factors affecting university admission using a bipolar fuzzy MCDM and also introduced a new algorithm that encompasses bipolar fuzzy sets and soft expert sets to improve the precision of decision-making within the education domain. Gul [32] introduced a new method for MCDM, by combining the bipolar fuzzy preference δ-covering based rough set model (BFPδC-BFRS) with the VIKOR approach to manage situations with positive and negative attributes and provided a case study of a real-world situation as well as a comparison with existing methods to demonstrate the efficacy and reliability of their proposed model.

Besides MCDM, a lot of research has been done on algebraic models under bipolar fuzzy conditions. Khan et al. [33] proposed and studied bipolar picture fuzzy graphs by defining their definitions, key characteristics, operations, and structural elements such as paths, degrees, and isomorphisms and also applied these concepts to generate a bipolar picture fuzzy acquaintanceship graph in order to study symmetry in both social and computer networks. Akmar [34] defined bipolar fuzzy graphs (BFGs) and strong BFGs, described various methods of their construction, discussed the concept of isomorphism of these graphs, investigated some of their important properties, and discussed some propositions of self-complementary and self-weak complementary strong BFGs. Akram et al. [35] developed iterative solution methods for bipolar fuzzy linear systems of equations (BFLSEs) using methods like Richardson, extrapolated Richardson, Jacobi, Jacobi over-relaxation, Gauss-Seidel, extrapolated Gauss-Seidel, and successive over-relaxation. Akram et al. [36] extended this work to fully bipolar fuzzy linear systems (FBFLSEs), proposing a novel solution technique utilizing (-1,1)-cut expansions to compute the extrema of all symmetric solutions for a given set of tolerable and controllable solution bounds. Wang et al. [37] proposed a new idea of bipolar T-spherical fuzzy sets (BTSFS) which is a new combination of bipolar fuzzy sets and T-spherical fuzzy sets, and logically investigates their basic properties and operations, and also new distance and similarity measures. It introduces the application of the measures to pattern recognition and creates a method for MCDM problem to show the practical usefulness of BTSFS. The introduction of bipolar complex fuzzy near rings (BCFNR) expands the algebraic framework for decision-making, accommodating inherent ambiguity and bipolarity [38]. Fuzzy bipolar hypersoft sets (FBHS) integrate bipolarity with fuzziness, providing a robust framework for modeling complex scenarios [39]. Compared to traditional fuzzy sets, BFSs and their extensions offer a more nuanced approach to uncertainty, making them superior for complex decision-making scenarios [7]. Collectively, these contributions illustrate that BFSs are evolving into a deeper level of sophistication and workplace applicability over an increasingly expansive area of influence in multiple fields such as project selection, environmental planning, healthcare diagnostics, and mathematical modeling.

3. Bipolar Fuzzy Sets

This section discusses about basic definitions of fuzzy set, BFS, trapezoidal bipolar fuzzy numbers and its arithmetic properties. Also, a new ranking function is defined and deeply studies its properties.

Definition 1 (fuzzy set [1]). The fuzzy set (FS) $\hat{A}$ in a universal set Ω is defined by

$$\widehat{A} = \{<x, \mu_A> : x \in \Omega\}, \tag{1}$$

the map $\mu_A : \Omega \to [0,1]$ is the membership grade.

Definition 2 (bipolar fuzzy set [4]). The BFS $\hat{B}$ in Ω is defined by

$$\widehat{B} = \{<x, \mu_B^+, \mu_B^-> : x \in \Omega\}, \tag{2}$$

The maps $\mu_B^+ : \Omega \to [0,1]$ and $\mu_B^- : \Omega \to [-1,0]$ are the positive and negative membership grades.

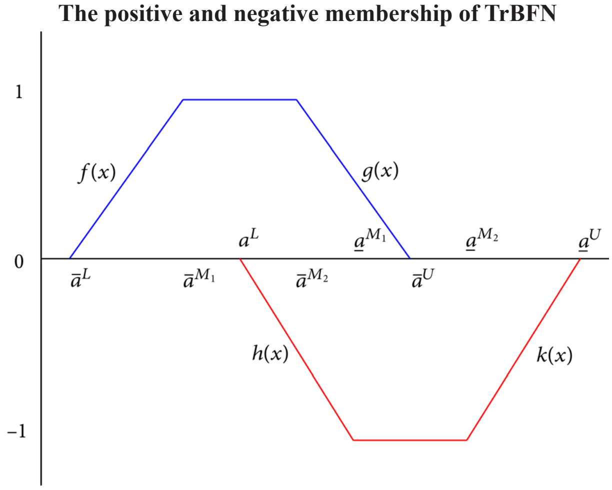
Definition 3 [40]. A trapezoidal bipolar fuzzy number (TrBFN) is denoted by $\widehat{A} = <\bar{a}^L, \bar{a}^{M_1}, \bar{a}^{M_2}, \bar{a}^U; \underline{a}^L, \underline{a}^{M_1}, \underline{a}^{M_2}, \underline{a}^U>$, and the positive and negative membership degrees as

$$\mu_A^+(x) = \begin{cases} f(x) = \frac{x - \bar{a}^L}{\bar{a}^{M_1} - \bar{a}^L}, & \bar{a}^L \le x \le \bar{a}^{M_1} \\ 1, & \bar{a}^{M_1} \le x \le \bar{a}^{M_2} \\ g(x) = \frac{\bar{a}^U - x}{\bar{a}^U - \bar{a}^{M_2}}, & \bar{a}^{M_2} \le x \le \bar{a}^U \\ 0, & Otherwise \end{cases} \tag{3}$$

$$\mu_A^-(x) = \begin{cases} h(x) = \frac{\underline{a}^L - x}{\underline{a}^{M_1} - \underline{a}^L}, & \underline{a}^L \le x \le \underline{a}^{M_1} \\ -1, & \underline{a}^{M_1} \le x \le \underline{a}^{M_2} \\ k(x) = \frac{x - \underline{a}^U}{\underline{a}^U - \underline{a}^{M_2}}, & \underline{a}^{M_2} \le x \le \underline{a}^U \\ 0, & Otherwise \end{cases} \tag{4}$$

Graphical representation of TrBFN is shown in Figure 1.

Figure 1
The positive and negative membership of TrBFN



Definition 4 (arithmetic properties [5]). Let Â and B̂ are the two TrBFNs, denoted by $\widehat{A} = \langle \bar{a}^L, \bar{a}^{M_1}, \bar{a}^{M_2}, \bar{a}^U; \underline{a}^L, \underline{a}^{M_1}, \underline{a}^{M_2}, \underline{a}^U \rangle$ and $\widehat{B} = \left\langle \bar{b}^L, \bar{b}^{M_1}, \bar{b}^{M_2}, \bar{b}^U; \underline{b}^L, \underline{b}^{M_1}, \underline{b}^{M_2}, \underline{b}^U \right\rangle$. The arithmetic properties of the TrBFNs are defined by

1) $\widehat{A} \oplus \widehat{B} = \left\langle \begin{matrix} \bar{a}^L + \bar{b}^L, \bar{a}^{M_1} + \bar{b}^{M_1}, \bar{a}^{M_2} + \bar{b}^{M_2}, \bar{a}^U + \bar{b}^U; \\ \underline{a}^L + \underline{b}^L, \underline{a}^{M_1} + \underline{b}^{M_1}, \underline{a}^{M_2} + \underline{b}^{M_2}, \underline{a}^U + \underline{b}^U \end{matrix} \right\rangle$

2) $\widehat{A} \ominus \widehat{B} = \left\langle \begin{matrix} \bar{a}^L - \bar{b}^U, \bar{a}^{M_1} - \bar{b}^{M_2}, \bar{a}^{M_2} - \bar{b}^{M_1}, \bar{a}^U - \bar{b}^L; \\ \underline{a}^L - \underline{b}^U, \underline{a}^{M_1} - \underline{b}^{M_2}, \underline{a}^{M_2} - \underline{b}^{M_1}, \underline{a}^U - \underline{b}^L \end{matrix} \right\rangle$

3) $\widehat{A} \otimes \widehat{B} = \left\langle \begin{matrix} \bar{a}^L \bar{b}^L, \bar{a}^{M_1} \bar{b}^{M_1}, \bar{a}^{M_2} \bar{b}^{M_2}, \bar{a}^U \bar{b}^L; \\ \underline{a}^L \underline{b}^L, \underline{a}^{M_1} \underline{b}^{M_1}, \underline{a}^{M_2} \underline{b}^{M_2}, \underline{a}^U \underline{b}^U \end{matrix} \right\rangle$

4) $\lambda \widehat{A} = \begin{cases} \left\langle \begin{matrix} \lambda \bar{a}^L, \lambda \bar{a}^{M_1}, \lambda \bar{a}^{M_2}, \lambda \bar{a}^U; \\ \lambda \underline{a}^L, \lambda \underline{a}^{M_1}, \lambda \underline{a}^{M_2}, \lambda \underline{a}^U \end{matrix} \right\rangle, & if\ \lambda \ge 0 \\ \left\langle \begin{matrix} \lambda \bar{a}^U, \lambda \bar{a}^{M_2}, \lambda \bar{a}^{M_1}, \lambda \bar{a}^L; \\ \lambda \underline{a}^U, \lambda \underline{a}^{M_2}, \lambda \underline{a}^{M_1}, \lambda \underline{a}^L \end{matrix} \right\rangle, & if\ \lambda \le 0 \end{cases}$

5) $\frac{\widehat{A}}{\widehat{B}} = \left\langle \frac{\bar{a}^L}{\bar{b}^U}, \frac{\bar{a}^{M_1}}{\bar{b}^{M_2}}, \frac{\bar{a}^{M_2}}{\bar{b}^{M_1}}, \frac{\bar{a}^U}{\bar{b}^L}, \frac{\underline{a}^L}{\underline{b}^U}, \frac{\underline{a}^{M_1}}{\underline{b}^{M_2}}, \frac{\underline{a}^{M_1}}{\underline{b}^{M_2}}, \frac{\underline{a}^{M_2}}{\underline{b}^{M_1}}, \frac{\underline{a}^U}{\underline{b}^L} \right\rangle$

Definition 5. The left and right magnitude of a TrBFN $\widehat{A} = \langle \bar{a}^L, \bar{a}^{M_1}, \bar{a}^{M_2}, \bar{a}^U; \underline{a}^L, \underline{a}^{M_1}, \underline{a}^{M_2}, \underline{a}^U \rangle$ is a crisp interval MGI(Â), is defined by

$$MGI\left(\widehat{A}\right) = \left[MG_L\left(\widehat{A}\right), MG_R\left(\widehat{A}\right)\right], \tag{5}$$

Where:

$$MG_L\left(\widehat{A}\right) = \frac{\bar{a}^{M_1} + \underline{a}^L}{2} + \frac{1}{2}\left[\int_{\underline{a}^L}^{\underline{a}^{M_1}} h(x)dx - \int_{\bar{a}^L}^{\bar{a}^{M_1}} f(x)dx\right]$$

$$MG_R\left(\widehat{A}\right) = \frac{\bar{a}^{M_2} + \underline{a}^U}{2} + \frac{1}{2}\left[\int_{\bar{a}^{M_2}}^{\bar{a}^U} g(x)dx - \int_{\underline{a}^{M_2}}^{\underline{a}^U} k(x)dx\right]$$

Definition 6. The magnitude of $\hat{A}$ is defined as average of lower and upper bound of $MGI(\hat{A})$. That is

$$MGI\left(\widehat{A}\right) = \frac{MG_L\left(\widehat{A}\right) + MG_R\left(\widehat{A}\right)}{2} \tag{6}$$

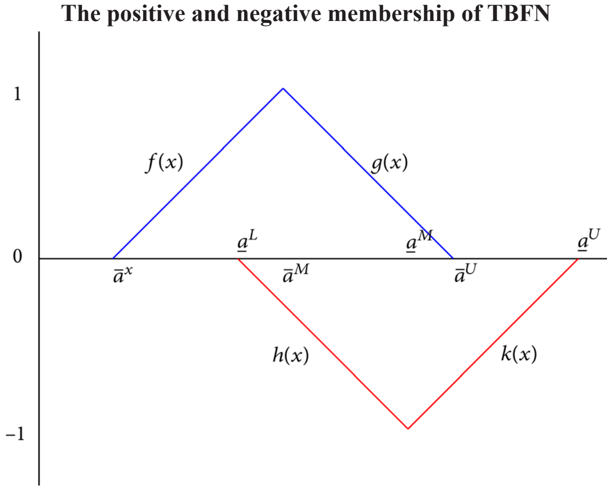
Definition 7 (Triangular bipolar fuzzy number [5]). A triangular bipolar fuzzy number (TBFN) $\hat{A}$ is created by considering $\bar{a}^{M_1} = \bar{a}^{M_2} = \bar{a}^M$ and $\underline{a}^{M_1} = \underline{a}^{M_2} = \underline{a}^M$ in a TrBFN $\hat{A}$ and it is denoted by $\widehat{A} = <\bar{a}^L, \bar{a}^M, \bar{a}^U; \underline{a}^L, \underline{a}^M, \underline{a}^U>$.The positive and negative membership grades are defined as

$$\mu_A^+(x) = \begin{cases} f(x) = \frac{x - \bar{a}^L}{\bar{a}^M - \bar{a}^L}, & \bar{a}^L \le x \le \bar{a}^M \\ g(x) = \frac{\bar{a}^U - x}{\bar{a}^U - \bar{a}^M}, & \bar{a}^M \le x \le \bar{a}^U \\ 0 & Otherwise \end{cases} \tag{7}$$

$$\mu_A^-(x) = \begin{cases} h(x) = \frac{\underline{a}^L - x}{\underline{a}^M - \underline{a}^L}, & \underline{a}^L \le x \le \underline{a}^M \\ k(x) = \frac{x - \underline{a}^U}{\underline{a}^U - \underline{a}^M}, & \underline{a}^M \le x \le \underline{a}^U \\ 0 & Otherwise \end{cases} \tag{8}$$

Graphical representation of TBFN is shown in Figure 2.

Figure 2
The positive and negative membership of TBFN



Definition 8. The magnitude for TBFN Â is calculated as

$$MG\left(\widehat{A}\right) = \frac{\bar{a}^L + 2\bar{a}^M + \bar{a}^U + \underline{a}^L + 2\underline{a}^M + \underline{a}^U}{8} \tag{9}$$

and the interval of left and right magnitude MGI(Â) is calculated as

$$MGI\left(\widehat{A}\right) = \left[\frac{\bar{a}^L + \bar{a}^M + \underline{a}^L + \underline{a}^M}{4}, \frac{\bar{a}^M + \bar{a}^U + \underline{a}^M + \underline{a}^U}{4}\right] \tag{10}$$

Definition 9 (ordering of TBFN). Let Â and B̂ are the two TBFN.

1) If $\hat{A} \le \hat{B}$ then $MG(\hat{A}) \le MG(\hat{B})$.
2) If $\hat{A} \ge \hat{B}$ then $MG(\hat{A}) \ge MG(\hat{B})$.
3) If $\hat{A} = \hat{B}$ then $MG(\hat{A}) = MG(\hat{B})$.

Example 3.1. Let $\hat{A}$ = ⟨31,38,43; 36,45,50⟩ and $\hat{B}$= ⟨35,39,46; 34,42,48⟩ are two TBFNs. Since, $MG(\hat{A}) = 40.75$ and $MG(\hat{B}) = 40.625$ then by using Definition 9, we have $\hat{A} > \hat{B}$.

Theorem 1. Let $\widehat{A_i} = \langle \bar{a}_i^L, \bar{a}_i^M, \bar{a}_i^U; \underline{a}_i^L, \underline{a}_i^M, \underline{a}_i^U \rangle$, for $i = 1, \ldots, n$ be the n TBFN in ***R***. Then

$$MG\left(\sum_{i=1}^{n} \widehat{A_i}\right) = \sum_{i=1}^{n} MG\left(\widehat{A_i}\right) \tag{11}$$

Corollary 1. Let $\hat{A}$ and $\hat{B}$ be the two TBFNs in R and $\lambda \in R$ be a real number, then

$$MG\left(\widehat{A} + \lambda\widehat{B}\right) = MG\left(\widehat{A}\right) + \lambda MG\left(\widehat{B}\right) \tag{12}$$

4. Bipolar Fuzzy Data Envelopment Analysis (BF-DEA)

Let there be n DMUs. Each DMU_j is characterized by an input vector $x \in R^m$ and an output vector $y \in R^r$. The data for all DMUs are aggregated into the input matrix $X = [x_1, \cdots, x_m] \in R^{m \times n}$ and the output matrix $Y = [y_1, \cdots, y_r] \in R^{r \times n}$, with the assumption that $X > 0$ and $Y > 0$. Charnes et al. [10] developed this model for measuring the efficiency of DMU_o i.e.,

$$\begin{aligned} \max_{u,v} \quad & \theta_o = \frac{\sum_{k=1}^{r} u_k y_{ko}}{\sum_{i=1}^{m} v_i x_{io}}, \\ \text{subject to} \quad & \frac{\sum_{k=1}^{r} u_k y_{kj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1, 2, \cdots n, \\ \text{and} \quad & u_k \ge 0, \quad k = 1, 2, \cdots, r, \\ & v_i \ge 0, \quad i = 1, 2, \cdots, m, \end{aligned} \tag{13}$$

The corresponding linear program (LP_o) is,

$$\begin{aligned} \max_{u,v} \quad & \theta_o = \sum_{k=1}^{r} u_k y_{ko}, \\ \text{subject to} \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\ \text{and} \quad & \sum_{k=1}^{r} u_k y_{kj} \le \sum_{i=1}^{m} v_i x_{ij} \quad j = 1, 2, \cdots n, \\ & u_k \ge 0, \qquad k = 1, 2, \cdots, r, \\ & v_i \ge 0, \qquad i = 1, 2, \cdots, m, \end{aligned} \tag{14}$$

solve the above LPP to find the optimal solution for DMU_o,($o = 1, 2, \ldots, n$) i.e., $\left(\theta_o^*, u_k^*, v_i^*\right)$. The efficiency score of the DMUo is θ_o^*.

Definition 10. A DMU is said to be efficient, if its efficiency score is one; otherwise, it is called as inefficient DMU.

Suppose the observed inputs and outputs are in TBFNs. The BFDEA model is defined as

$$\max_{u,v} \quad \theta_o = \sum_{k=1}^{r} u_k \widehat{y_{ko}}, \tag{15}$$
$$\text{subject to} \quad \sum_{i=1}^{m} v_i \widehat{x_{io}} = 1,$$
$$\text{and} \quad \begin{aligned} &\sum_{k=1}^{r} u_k \widehat{y_{kj}} \le \sum_{i=1}^{m} v_i \widehat{x_{ij}} && j = 1, 2, \cdots n, \\ &u_k \ge 0, && k = 1, 2, \cdots, r, \\ &v_i \ge 0, && i = 1, 2, \cdots, m, \end{aligned}$$

where the inputs $\widehat{x_{ij}} = \left\langle \overline{x_{ij}}^L, \overline{x_{ij}}^M, \overline{x_{ij}}^U; \underline{x_{ij}}^L, \underline{x_{ij}}^M, \underline{x_{ij}}^U \right\rangle$ and outputs $\widehat{y_{kj}} = \left\langle \overline{y_{kj}}^L, \overline{y_{kj}}^M, \overline{y_{kj}}^U; \underline{y_{kj}}^L, \underline{y_{kj}}^M, \underline{y_{kj}}^U \right\rangle$ are TBFNs, and the weights are crisp real number.

Apply the magnitude function (MG) in the BFDEA model, we have

$$\max_{u,v} \quad \theta_o = \mathrm{MG}\left(\sum_{k=1}^{r} u_k y_{ko}\right), \tag{16}$$
$$\text{subject to} \quad MG\left(\sum_{i=1}^{m} v_i x_{io}\right) = 1,$$
$$\text{and} \quad \begin{aligned} &MG\left(\sum_{k=1}^{r} u_k y_{kj}\right) \le \mathrm{MG}\left(\sum_{i=1}^{m} v_i x_{ij}\right) && j = 1, 2, \cdots n, \\ &u_k \ge 0, && k = 1, 2, \cdots, r, \\ &v_i \ge 0, && i = 1, 2, \cdots, m, \end{aligned}$$

Applying Theorem 1 and Corollary 1, we have

$$\max_{u,v} \quad \theta_o = \sum_{k=1}^{r} \left(\frac{\overline{y_{ko}}^L + 2\overline{y_{ko}}^M + \overline{y_{ko}}^U + \underline{y_{ko}}^L + 2\underline{y_{ko}}^M + \underline{y_{ko}}^U}{8} \right) u_k \tag{17}$$
$$\text{s.t.} \quad \sum_{i=1}^{m} \left(\frac{\overline{x_{io}}^L + 2\overline{x_{io}}^M + \overline{x_{io}}^U + \underline{x_{io}}^L + 2\underline{x_{io}}^M + \underline{x_{io}}^U}{8} \right) v_i x_{io} = 1,$$
$$\sum_{k=1}^{r} \left(\frac{\overline{y_{kj}}^L + 2\overline{y_{kj}}^M + \overline{y_{kj}}^U + \underline{y_{kj}}^L + 2\underline{y_{kj}}^M + \underline{y_{kj}}^U}{8} \right) u_k$$
$$\le \sum_{i=1}^{m} \left(\frac{\overline{x_{ij}}^L + 2\overline{x_{ij}}^M + \overline{x_{ij}}^U + \underline{x_{ij}}^L + 2\underline{x_{ij}}^M + \underline{x_{ij}}^U}{8} \right) v_i, \quad j = 1, 2, \cdots n$$
$$\text{and} \quad u_k \ge 0, \quad k = 1, 2, \cdots, r; \quad v_i \ge 0, \quad i = 1, 2, \cdots, m,$$

which is the crisp LP model of the proposed BFDEA model.

4.1. Complete ranking technique

The DEA literature faces the challenge of ranking efficient DMUs with a unity score. Andersen and Petersen [12] developed the super-efficiency model, which ranks efficient DMUs based on their super efficiency score. DMUs are entirely ranked according to their super efficiency score. This traditional model can be extended to fuzzy model by considering the inputs and outputs are fuzzy numbers. The bipolar fuzzy super-efficiency model is presented by assuming the inputs and outputs are bipolar fuzzy numbers for assessing super efficiency score of the DMUs in the bipolar fuzzy environment, which is described as

$$\max_{u,v} \quad \theta_o = \sum_{k=1}^{r} u_k \widehat{y_{ko}}, \tag{18}$$
$$\text{subject to} \quad \sum_{i=1}^{m} v_i \widehat{x_{io}} = 1,$$
$$\text{and} \quad \begin{aligned} &\sum_{k=1}^{r} u_k \widehat{y_{kj}} \le \sum_{i=1}^{m} v_i \widehat{x_{ij}} && j = 1, 2, \cdots n, j \ne o \\ &u_k \ge 0, && k = 1, 2, \cdots, r, \\ &v_i \ge 0, && i = 1, 2, \cdots, m, \end{aligned}$$

Applying the solution procedure for BFDEA model to find the corresponding crisp super efficiency model is given as

$$\max_{u,v} \quad \theta_o = \sum_{k=1}^{r} \left(\frac{\overline{y_{ko}}^L + 2\overline{y_{ko}}^M + \overline{y_{ko}}^U + \underline{y_{ko}}^L + 2\underline{y_{ko}}^M + \underline{y_{ko}}^U}{8} \right) u_k \tag{19}$$
$$\text{s.t.} \quad \sum_{i=1}^{m} \left(\frac{\overline{x_{io}}^L + 2\overline{x_{io}}^M + \overline{x_{io}}^U + \underline{x_{io}}^L + 2\underline{x_{io}}^M + \underline{x_{io}}^U}{8} \right) v_i x_{io} = 1,$$
$$\sum_{k=1}^{r} \left(\frac{\overline{y_{kj}}^L + 2\overline{y_{kj}}^M + \overline{y_{kj}}^U + \underline{y_{kj}}^L + 2\underline{y_{kj}}^M + \underline{y_{kj}}^U}{8} \right) u_k$$
$$\le \sum_{i=1}^{m} \left(\frac{\overline{x_{ij}}^L + 2\overline{x_{ij}}^M + \overline{x_{ij}}^U + \underline{x_{ij}}^L + 2\underline{x_{ij}}^M + \underline{x_{ij}}^U}{8} \right) v_{i,} \quad j = 1, 2, \cdots, n, \quad j \ne 0$$
$$\text{and} \quad u_k \ge 0, \quad k = 1, 2, \cdots, r; \quad v_i \ge 0, \quad i = 1, 2, \cdots, m,$$

which is the crisp LP model of the proposed Bipolar fuzzy super efficiency model.

4.2. Benchmarking technique

Benchmarking in a DEA framework involves defining targets and identifying a peer group as a reference set for the projection point on the efficient frontier. Targets are the coordinates of a unit's projection point on the efficient frontier, obtained by combining efficient DMUs on a face of the frontier. These efficient DMUs are commonly categorized into "reference sets." Equation (20) provides the reference set or peers group for each DMU in the BF-DEA model. The reference set for inefficient DMUs provides a collection of DMUs that serve as benchmarks for the target inefficient DMUs. Let us consider the set of such *DMUo*, $o \in \{1, \cdots, n\}$ be

$$E_o = \left\{ DMU_j : \sum_{k=1}^{r} \left(\frac{\overline{y_{kj}}^L + 2\overline{y_{kj}}^M + \overline{y_{kj}}^U + \underline{y_{kj}}^L + 2\underline{y_{kj}}^M + \underline{y_{kj}}^U}{8} \right) u_k^* \right.$$
$$= \sum_{i=1}^{m} \left(\frac{\overline{x_{ij}}^L + 2\overline{x_{ij}}^M + \overline{x_{ij}}^U + \underline{y_{ij}}^L + 2\underline{x_{ij}}^M + \underline{x_{ij}}^U}{8} \right) v_i^*, \quad j \tag{20}$$
$$\left. = 1, 2, \cdots, n, \right\}$$

The reference set (or peer group) for a specific unit, DMU_o, is defined as the set E_o, comprising efficient DMUs whose performance benchmarks reveal the inefficiency of DMU_o. The efficient frontier relevant to DMU_o is the facet of the production frontier constructed by the linear combination of the units within E_o.

5. Solution Procedure

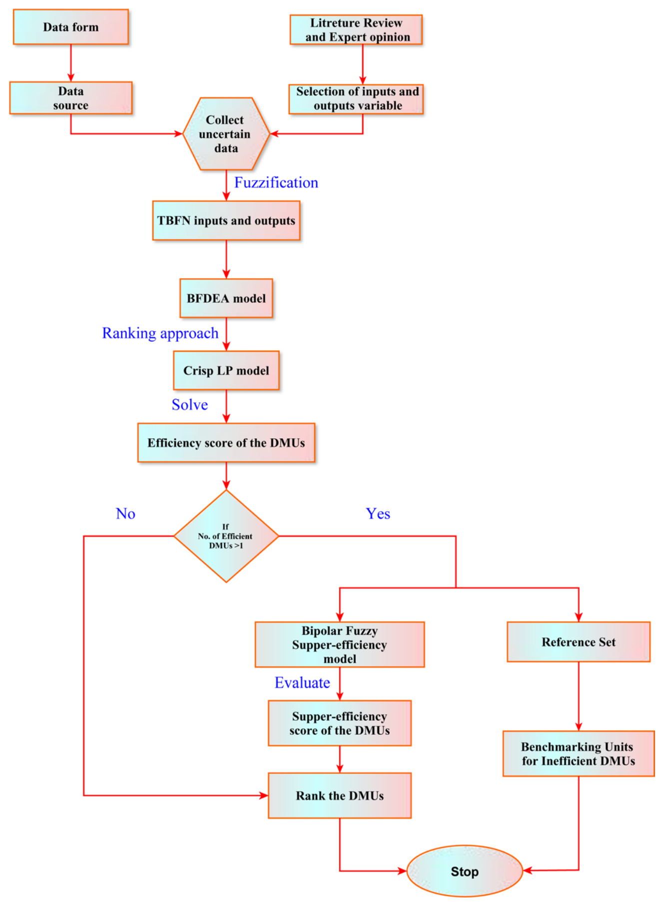
The proposed algorithm assesses the efficiency of the DMUs within a bipolar fuzzy milieu utilizing DEA. It commences by processing fuzzy inputs and outputs for all DMUs, ensuring the algorithm accommodates ambiguity and inexactitude in empirical data. The algorithm repetitively implements the BF-DEA model to compute the relative efficiency indices of DMUs, distinguishing efficient from inefficient units. In scenarios involving multiple efficient DMUs, the BF-Super-efficiency model is utilized, facilitating more nuanced differentiations among high performing units by allocating super-efficiency scores. This guarantees a resilient ranking mechanism even for the efficient DMUs. Furthermore, the algorithm delineates reference sets and benchmarking units, providing actionable insights for enhancing the performance of inefficient DMUs. By amalgamating efficiency assessment, ranking, and benchmarking, the Algorithm 1 offers a comprehensive framework for performance analysis in intricate environments where data manifests bipolar fuzzy characteristics. This algorithm implemented in MATLAB R2015a software the solve the proposed BFDEA model and measured the efficiency, ranking and benchmarking of the DMUs.

Figure 3 illustrates the step-by-step solution approach for measuring the performance of DMUs in a bipolar fuzzy environment.

Algorithm 1 Algorithm for measuring DMUs' performance in Bipolar fuzzy environment

Require: Bipolar Fuzzy Inputs and Outputs
Ensure: Efficiency score, Ranking, Benchmarking
X ← Inputs
Y ← Outputs
n ← Number of DMUs
while $n \neq 0$ do
  Solve BF-DEA model using X & Y.
Evaluate Efficiency Score of each DMU.
if (No. of Efficient DMU > 1) then
Solve the BF-Super-efficiency model
Evaluate Super-efficiency score of each DMU.
Rank the DMUs based on super-efficiency score. AND
Evaluate the Reference Set for each DMU.
Calculate the benchmarking units for each DMU.
end if
Rank the DMUs based on their relative efficiency score.
end while

Figure 3
Solution technique for the performance analysis in bipolar fuzzy environment



6. Numerical Example

A numerical example is presented herein to elucidate the validity and applicability of the suggested performance assessment methodologies within a bipolar fuzzy context. For this illustration, we examine a scenario involving ten DMUs characterized by two input parameters and two output variables, which are represented as triangular bipolar fuzzy numbers as shown in Table 2.

The efficiency evaluation using BFDEA model demonstrates a clear distinction between efficient and inefficient DMUs. Table 3 presents the efficiency scores, where D1, D5, D7, and D9 are treated as efficient with highest efficiency score of 1.0000. These DMUs are situated on the efficiency frontier, signifying optimal performance in resource utilization and output production. In contrast, D2, D3, D4, D6, D8, and D10 are classified as inefficient, with efficiency scores less than 1, indicating varying levels of suboptimal performance. Figure 4 further illustrates the efficiency scores of all DMUs, providing a comparative visualization of their relative efficiency scores.

The super-efficiency scores of all the DMU, as shown in Table 3, provide additional granularity in ranking their performance. Among the efficient DMUs, D5 achieves the highest super-efficiency score of 1.4625, establishing its position as the best-performing unit, followed by D9 (1.4045), D7 (1.3822), and D1 (1.1213). For the inefficient DMUs, their rankings align with their efficiency scores, with D2 being the least inefficient and D8 requiring the most improvement to reach the efficiency frontier.

Table 2
The triangular bipolar fuzzy inputs and outputs data

DMU	Input 1	Input 2	Output 1	Output 2
D1	(54, 61, 69; 47, 52, 58)	(86, 92, 97; 84, 87, 91)	(42, 47, 53; 40, 45, 50)	(23, 27, 30; 21, 23, 27)
D2	(71, 78, 83; 68, 74, 79)	(51, 56, 64; 54, 58, 63)	(34, 40, 45; 36, 45, 49)	(17, 20, 24; 16, 22, 25)
D3	(104, 108, 115; 105, 111, 118)	(43, 48, 55; 41, 46, 51)	(20, 26, 33; 18, 23, 27)	(9, 12, 18; 8, 13, 16)
D4	(64, 67, 72; 55, 58, 65)	(74, 79, 88; 69, 75, 80)	(55, 59, 65; 51, 54, 61)	(11, 14, 18; 12, 16, 19)
D5	(45, 57, 67; 49, 55, 60)	(47, 53, 57; 43, 46, 50)	(66, 73, 78; 57, 63, 69)	(7, 11, 16; 9, 13, 17)
D6	(120, 128, 135; 111, 117, 126)	(35, 39, 46; 35, 42, 48)	(25, 31, 37; 21, 26, 31)	(20, 23, 25; 18, 21, 22)
D7	(31, 38, 43; 36, 45, 50)	(61, 68, 74; 58, 63, 69)	(57, 64, 70; 52, 58, 66)	(13, 15, 18; 14, 17, 21)
D8	(84, 89, 95; 81, 87, 91)	(49, 56, 61; 52, 58, 63)	(33, 37, 44; 31, 36, 39)	(7, 10, 13; 6, 8, 11)
D9	(116, 120, 127; 108, 114, 119)	(31, 36, 43; 25, 30, 36)	(21, 26, 32; 24, 28, 32)	(24, 26, 29; 21, 23, 27)
D10	(87, 91, 98; 85, 90, 95)	(42, 48, 55; 39, 45, 50)	(37, 43, 49; 40, 45, 50)	(10, 12, 16; 7, 11, 14)

Table 3
Efficiency score and complete ranking of the DMUs

DMUs	Efficiency score	Type	Super efficiency score	Ranking
D1	1	Efficient	1.1213	4
D2	0.9536	Inefficient	0.9536	5
D3	0.5183	Inefficient	0.5183	9
D4	0.731	Inefficient	0.731	8
D5	1	Efficient	1.4625	1
D6	0.8105	Inefficient	0.8105	6
D7	1	Efficient	1.3822	3
D8	0.4996	Inefficient	0.4996	10
D9	1	Efficient	1.4045	2
D10	0.7439	Inefficient	0.7439	7

Figure 4
Comparison of efficiency score of the DMUs

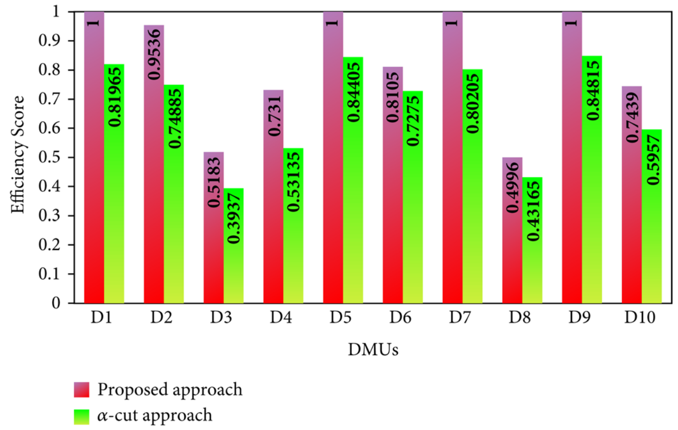


Table 5
Benchmarking units for the DMUs

DMUs	Benchmarking units
D1	D1
D2	D1, D7, D9
D3	D5, D7, D9
D4	D5, D7, D9
D5	D5
D6	D5, D7, D9
D7	D7
D8	D5, D9
D9	D9
D10	D5, D9

The efficiency score of each DMU is calculated using α-cut approach for fuzzy DEA model, shown in Table 4 [41]. The TBFN-inputs and outputs shown in Table 2 are converted into fuzzy inputs and outputs by eliminating the parameters associated with negative membership degree. The efficiency score in Table 4 shows the results in interval form based on α-cut approach that is not easy to identify the best performer DMUs among them. The mean efficiency score is a solution for this problem but this approach cannot find which DMUs can be categorized into efficient or inefficient group. But the proposed bipolar fuzzy DEA is presented the efficiency in exact level and it also analysis each DMUs into efficient and inefficient. To rank the DMUs the proposed bipolar fuzzy super efficiency model is used to completely ranked the DMUs in that the best efficient DMU identified among the efficient DMUs. However, the proposed BFDEA model have the capacity to identify the benchmarking for inefficient DMUs, which aids in the identification areas of improvement for decision makers.

The comprehensive benchmarking analysis presented in Table 5 clearly indicates that DMUs D1, D5, D7, and D9 stand out as exemplary reference units, given that they serve as benchmarks for an array of less efficient DMUs, thereby contributing significantly to the overall understanding of operational efficiency within the assessed framework. In more specific terms, the less efficient DMUs, namely D2, D3, D4, D6, D8, and D10, exhibit a reliance on the efficient units mentioned earlier, utilizing these benchmarks to inform and catalyze their respective performance enhancement strategies. Among the aforementioned reference units, it is particularly noteworthy that D5 and D9 have emerged as the most frequently cited benchmarks, which serves to underscore their robust operational efficiency and effectiveness in their respective functions. Furthermore, the fact that DMUs D1, D5, D7, and D9 are identified as self-benchmarked reinforces their preeminent status on the efficiency frontier, suggesting that these units exemplify optimal performance standards within the operational landscape. In stark contrast, the inefficient DMUs such as D2, D3, D4, D6, D8, and D10 face the imperative of recalibrating their input-output ratios in accordance with the benchmarks set by their more efficient counterparts, a necessary adjustment aimed at fostering enhanced efficiency outcomes. The existence of multiple benchmarks for certain DMUs not only highlights the variability in pathways to achieving efficiency but also implies that these pathways may differ significantly based on the specific operational contexts and the strategic approaches to resource utilization that are employed. Thus, this in-depth analysis yields valuable and actionable insights that can facilitate performance improvement initiatives by pinpointing best practices derived from the operational excellence exhibited by their efficient peers. In summary, the findings from this benchmarking analysis not only contribute to the theoretical understanding of efficiency but also provide a practical framework for less efficient DMUs to emulate in their quest for enhanced operational performance. Ultimately, the identification of these benchmarks serves as a critical resource for organizations aiming to optimize their operational strategies and improve their overall efficiency metrics in a competitive landscape.

Table 4
Efficiency score in α-cut approach

DMUs	*α-cut* approach		
	α=0.5	Mean efficiency	Ranking
D1	[0.6393, 1]	0.81965	3
D2	[0.5392, 0.9585]	0.74885	5
D3	[0.2181, 0.5693]	0.3937	10
D4	[0.3884, 0.6743]	0.53135	8
D5	[0.6881, 1]	0.84405	2
D6	[0.5512, 0.9038]	0.7275	6
D7	[0.6041, 1]	0.80205	4
D8	[0.3023, 0.561]	0.43165	9
D9	[0.6963, 1]	0.84815	1
D10	[0.4106, 0.7808]	0.5957	7

The analysis underscores the value of bipolar fuzzy DEA in providing a detailed assessment of efficiency and performance rankings. Efficient DMUs serve as benchmarks, offering insights and strategies for the improvement of inefficient units. The combination of efficiency and super-efficiency scores facilitates a robust framework for performance comparison and decision-making across the DMUs.

7. Conclusion

The proposed BFDEA model serves as a highly effective tool for measuring the performance metrics of DMUs operating within environments that are inherently characterized by uncertainty and ambiguity, thereby facilitating a comprehensive understanding of their operational efficiencies. By integrating fuzzy inputs and outputs into its analytical framework, the model significantly enhances its robustness in accurately capturing the multifaceted complexities that are often present in real-world scenarios. The outcomes derived from this analytical approach reveal distinct and unequivocal differentiations between DMUs categorized as efficient and those identified as inefficient, with efficient units such as D5, D9, D7, and D1 attaining an exemplary efficiency score of 1.0000, thereby showcasing their operational effectiveness. Furthermore, the super-efficiency analysis, which is a critical component of this framework, positions D5 at the pinnacle of performance among the assessed DMUs, evidenced by its impressive score of 1.4625, thereby underscoring the BFDEA's adeptness in yielding nuanced and sophisticated performance differentiation. The benchmarking framework, which serves as an additional analytical layer, further assists inefficient DMUs in their pursuit of improvement by illuminating reference sets and best practices, thereby providing a strategic pathway for enhancing their operational efficiencies.

Future research might extend the use of the BF-DEA model to include multi-period or dynamic settings, allowing for the evaluation of DMU performance across time. Integrating machine learning approaches with BF-DEA may improve forecasting capacities and reveal patterns in efficiency changes under uncertain settings. Furthermore, investigating hybrid models that integrate fuzzy DEA with other MCDM approaches, such as TOPSIS or AHP, might result in an improved decision-support framework. Finally, applying the concept to other areas, such as healthcare, education, insurance, banking, supply chain management, manufacturing, and energy, might demonstrate its versatility and applicability across other disciplines.

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Ethical Statement

This study does not contain any studies with human or animal subjects performed by the author.

Conflicts of Interest

The author declares that he has no conflicts of interest to this work.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Author Contribution Statement

Kshitish Kumar Mohanta: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing – original draft, Writing – review & editing, Visualization, Supervision, Project administration.

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