Received: 27 October 2024 | Revised: 12 March 2025 | Accepted: 3 April 2025 | Published online: 7 May 2025

RESEARCH ARTICLE

Journal of Data Science and Intelligent Systems
2025, Vol. 00(00) 1–15
DOI: 10.47852/bonviewJDSIS52024661

BON VIEW PUBLISHING

Analytical Solution for Parameter Estimation of Weibull Distributions with Interval-Censored Data

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Abstract: This paper addresses the challenge of parameter estimation for Weibull distributions with continuous interval-censored data, a critical issue in reliability engineering where failures are observed only at predetermined inspection intervals. Traditional estimation methods often struggle with the uncertainty of failure times, leading to suboptimal results. To overcome this limitation, we propose a novel analytical approach that directly fits the probability density function to the frequency histogram, offering an alternative to conventional numerical algorithms. This method not only improves estimation accuracy but also enhances computational efficiency. Theoretical validation is established using the dual least squares method, and extensive Monte Carlo simulations further confirm its robustness. Comparative analysis with existing approaches highlights the superiority of our method in terms of both precision and stability. To demonstrate its practical applicability, we apply the proposed approach to Hong Kong casualty data from the World Health Organization, effectively estimating the age distribution of unidentified casualties. The results underscore the method's potential for broader applications in reliability analysis and risk assessment.

Keywords: interval-censored data, Weibull distribution, parameter estimation, analytical solution, lifetime distribution, weighted least squares method

1. Introduction

Reliability research is highly prevalent in the field of engineering, where various components, machines, and equipment require a reliable lifespan as a reference to ensure the safe and successful progression of projects. The lifespan is typically quantified using numerical probabilities [1]. With the continuous expansion of the theoretical depth of probability theory and statistics, the discovery of various distributions has provided powerful support for fitting data in real-life production and daily life. In reliability engineering and survival analysis, the Weibull distribution is frequently employed to model the lifespan distribution of various mechanical components, such as wind turbines, aircraft door lock mechanism, and computed tomography equipment [2–4]. Similarly, the two-parameter Weibull distribution plays a significant role in modeling processes, particularly in reflecting degradation rates [5–7]. In essence, it is utilized across diverse fields including materials science, engineering, physics, chemistry, meteorology, medicine, pharmacy, economics, business, quality control, biology, geology, and geography [8].

However, in practical engineering applications, factors such as high testing costs and stringent time constraints often serve as key limiting conditions that impede the accurate observation of product lifespan data. Affected by this, only the boundary values of the data can often be obtained, thus forming the interval-censored data that is the focus of this paper. Interval censoring means that the exact survival time of a product cannot be precisely known, and it is only clear that it lies within a certain specific time interval. For a more detailed discussion, refer to Section 3.

In this study, we apply the weighted least squares approach to continuous interval-censored data and derive a closed-form solution for parameter inference in the bivariate Weibull distribution through a series of rigorous mathematical derivations. This method helps us bypass the iterative solution phase of numerical algorithms (EM and MCMC [9, 10]), compensating for the deficiency of least squares estimation in providing a closed-form solution.

The structure of this paper is as follows: Section 2 summarizes the commonly used methods for Weibull parameter estimation, analyzes their advantages and disadvantages, and investigates existing approaches for handling interval-censored data. Section 3 provides a detailed introduction to the explicit solution for parameter estimation of continuous interval-censored data, and offers meaningful discussions on the analytical expressions. Section 4 first conducts Monte Carlo simulations to verify the effectiveness of the analytical solution method. Secondly, we compare it with other methods from the literature, including maximum likelihood estimation (MLE). Finally, the analytical solution method is extended to fixed-censoring tests and continuous sequential censoring tests, and an explanation is provided for the three-parameter case. Section 5 effectively applies this method using real data. Finally, Section 6 provides a summary of the paper and discusses the implications and potential extensions of the proposed method.

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2. Background for Research

If a random variable follows a Weibull distribution, its probability density function (PDF) can be expressed as follows:

$$f_T(t;\alpha,\lambda,\gamma)=\tfrac{\alpha}{\lambda}\left(\tfrac{t-\gamma}{\lambda}\right)^{\alpha-1}e^{-\left(\frac{t-\gamma}{\lambda}\right)^{\alpha}};t>0 \quad (1)$$

where α, λ, γ are represent the shape parameter, scale parameter, and location parameter, respectively. The form we are using here is the most common form of the Weibull distribution, and for more extended forms, one can refer to Lai et al. [11].

The different failure modes of a product are determined by the shape parameter α of the Weibull distribution. When $\alpha>1$, the failure rate increases over time, indicating a "wear-out" or "aging" phase where the likelihood of failure grows as the product gets older. This is often referred to as a "decreasing" or "increasing" failure rate model, depending on the context. When $\alpha=1$, the failure rate is constant over time, which corresponds to a "memoryless" property and is characteristic of an exponential distribution. This situation implies that the risk of failure does not change with age, which is often associated with "random" or "chance" failures that are not influenced by the product's age.

When $\alpha<1$, the failure rate decreases over time, suggesting that the product becomes more reliable as it ages. This is less common in practice and could be indicative of "infant mortality" or "burn-in" periods where initial failures are more likely, followed by a period of increased reliability. The scale parameter λ affects the time scale over which failures occur. The larger the value of λ, the longer it takes for failures to occur, and the wider the range over which the PDF is spread out. This parameter essentially shifts the PDF along the time axis, affecting the spread and scale of the distribution without changing its shape. In summary, the shape parameter λ dictates the type of failure pattern over time, while the scale parameter influences the time scale and spread of the distribution. The position parameter γ can also be referred to as the scale parameter.

By applying a translation transformation $\tilde{t}=t-\gamma$, Equation (1) becomes the pdf of the two-parameter Weibull distribution, as indicated in the Equation (2).

$$f_{\tilde{T}}(\tilde{t};\alpha,\lambda)=\frac{\alpha}{\lambda}\left(\frac{\tilde{t}}{\lambda}\right)^{\alpha-1}e^{-\left(\frac{\tilde{t}}{\lambda}\right)^{\alpha}} \quad (2)$$

When evaluating a product's reliability, we commonly use the reliability function, denoted as $R(t)$, also referred to as the survival function.

It represents the probability that a product will perform a specific function under given conditions and over a specified time period. Since "the product performs a specific function within time" is equivalent to "the product's lifespan T is greater than t", we can calculate the probability that the product's lifespan exceeds t using the following formula:

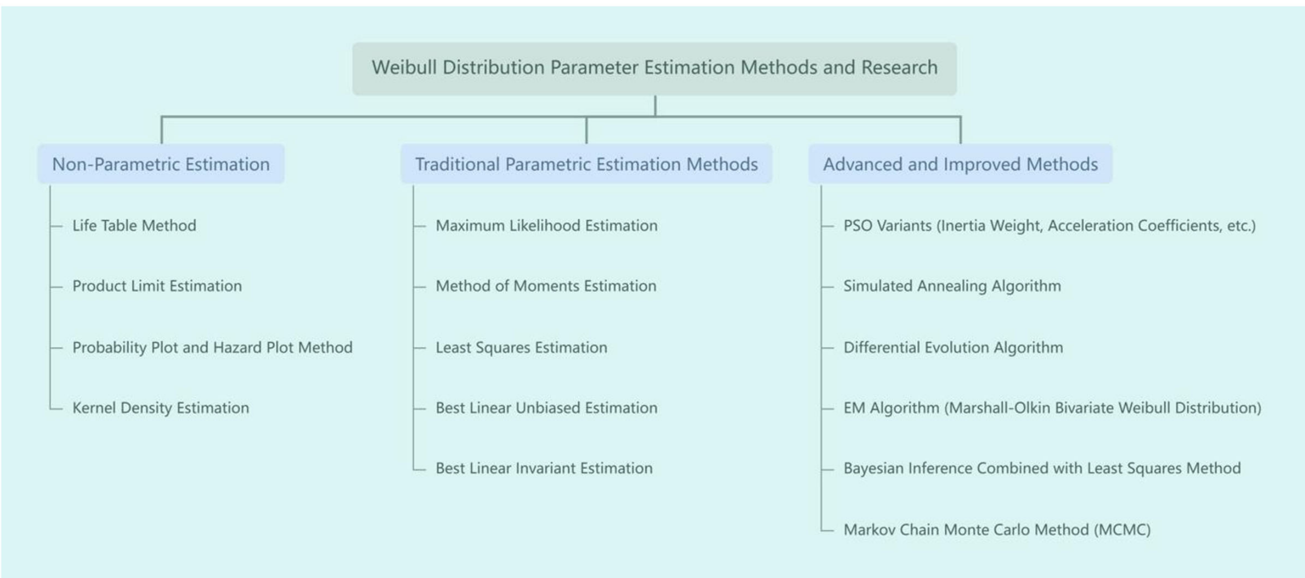
$$R(t)=P(T>t)=\int_t^{+\infty}f(x)dx=e^{-\left(\frac{t-\gamma}{\lambda}\right)^{\alpha}} \quad (3)$$

The failure rate of a product, denoted as $h(t)$, is defined as the probability that the product, which has been functioning up to a specific point in time, will fail within the subsequent time unit. The failure rate is a key measure in reliability engineering and is used to describe how the likelihood of failure changes over time. For a product that follows a Weibull distribution, the failure rate can be calculated using the following formula:

$$h(t)=\tfrac{\alpha}{\lambda^{\alpha}}(t-\gamma)^{\alpha-1} \quad (4)$$

For parameter estimation of the Weibull distribution, various methods can be utilized, as illustrated in Figure 1. Traditional methods include the life table method, product-limit estimation, and probability plotting with hazard plotting methods, among others. These are all non-parametric estimation techniques, which are advantageous due to their flexibility and lack of dependence on the data distribution. However, they may entail substantial computational effort, and the interpretation of the data may not be as intuitive as with parametric methods. In addition to these, there are some traditional parametric estimation methods, such as method of moments estimation, least squares estimation, best linear unbiased estimation (BLUE), and best linear unbiased

Figure 1
Classification diagram of parameter estimation methods for the Weibull distribution



invariant estimation. Parametric estimation typically offers precise predictions and inferences, with stronger interpretability.

However, in the actual process of parameter estimation, it has been observed that these estimation methods have significant limitations. The least squares method (LSM) involves transforming nonlinear models into linear ones, a process that alters the distribution type within the model, thus no longer satisfying the Gauss-Markov assumptions, leading to reduced estimation accuracy [12]. Moreover, the empirical distribution function used in LSM often relies on the median rank formula, which can also fail to fully adapt to the sample data.

Building upon traditional parameter estimation methods, numerous improvements and extensions have also been proposed, see Jia [13, 14] for details. The problem of maximizing the likelihood function in MLE has always been challenging. Kundu and Dey [9] used the EM algorithm to calculate the location parameters in the Marshall-Olkin bivariate Weibull distribution. Jiang et al. [10] suggested the Markov Chain Monte Carlo (MCMC) method for estimating the parameters of the modified Weibull distribution based on complete samples.

Parameter estimation is an indispensable part of statistical inference in reliability data analysis, yet understanding clearly the data we are dealing with is equally crucial. This paper focuses on censoring data, which often arises in applications across various fields such as epidemiology and medical research [15]. In medicine, the analysis of data that includes censored observations is referred to as survival analysis, which differs from reliability analysis typically conducted in industrial production. Such censoring data come in various forms, for instance, biomarker concentrations of interest in urine, serum, or other biological matrices [16], which can be handled using imputation methods, complete case analysis, and MLE [17]. In addition to these traditional methods, there are other approaches for parameter estimation. Spline functions can be employed for inference of left-truncated and right-truncated data [18], while penalized Cox proportional hazards models assess the impact of predictor variables on survival [19]. Penalized generalized empirical likelihood estimators are constructed for hypothesis testing in non-parametric likelihood missing survival model assumptions [20]. Furthermore, the types of missing data extend beyond the scope covered by definitions, necessitating different estimation methods for specific scenarios. For example, multi-stage SCAD penalized estimation equations are utilized for right-truncated length-biased data variable selection [21]. Non-parametric maximum likelihood joint modeling is employed for multivariate interval-censored survival data [22]. Semi-parametric mixed models are used for longitudinal missing data with Gaussian errors [23].

Although numerous effective parameter estimation methods have been proposed in existing studies, the estimation of Weibull distribution parameters under censored data universally faces three major challenges: First, numerical iterative optimization algorithms (e.g., EM and MCMC) suffer from exponentially increasing computational complexity, leading to inefficiency in large-scale data processing. Second, traditional parameter estimation methods (e.g., MLE) often encounter ill-conditioned likelihood functions in censored scenarios, resulting in significantly reduced estimation accuracy. Third, most existing approaches rely on numerical approximations and lack closed-form analytical solutions, which severely limits their applicability in engineering applications such as real-time monitoring systems. Therefore, proposing an analytical solution for Weibull distribution parameter estimation under censored data is both necessary and meaningful.

3. Analytical Solution of Parameter Estimation

Interval-censored data in continuous intervals have significant practical value, especially in situations where the exact failure times of products cannot be precisely observed. Defining intervals offers an effective solution in such cases. This study assumes that the lifetimes of n products follow a Weibull distribution. Typically, the experimental design requires setting predetermined inspection time points and recording the number of failed products at each point. However, tracking until the final product fails is often impractical because the process can be excessively time-consuming. Consequently, the experiment may end at a predetermined observation cutoff time.

A concrete example from everyday life can help illustrate the concept of interval-censored data more intuitively. Streetlights are susceptible to sudden failures, yet in practice, it is often difficult to pinpoint the exact time when each streetlight malfunctions. However, by inspecting the status of streetlights at specific time points, we can identify the time intervals during which failures occur most frequently. This approach significantly reduces labor and time costs, enabling efficient resource utilization.

To better illustrate this data, we use Table 1 for presentation.

Building on prior research, this section introduces a new method for parameter estimation using interval data. The core idea of this method is intuitive and easy to understand, resembling the empirical distribution function, which can effectively approximate the distribution function. By employing weighted first derivative values to fit the frequency distribution of the sample data, we derive an explicit estimation of the Weibull distribution parameters through meticulous and rigorous mathematical expressions.

In a non-strict sense, the law of large numbers allows us to estimate probabilities through frequencies. Therefore, we need to minimize the Q value in the following equation to the greatest extent.

$$Q = \sum_{i=1}^{r} \left[e^{-\left(\frac{t_i}{\lambda}\right)^{\alpha}} - \left(1 - \frac{\sum_{j=1}^{i} n_j}{n}\right) \right]^2 \tag{5}$$

Proposition 1. Given that $\mathrm{R_i} = \sum_{\mathrm{j}=1}^{\mathrm{i}} \mathrm{n_j}$, the explicit expression for the Weibull parameter estimation based on the interval-censored data from Table 1 is as follows, which also satisfies the condition of minimizing the Q value.

$$\widehat{\alpha} = \frac{\sum_{i=1}^{r} \alpha_i \, ln\left(-ln f_n(t_i)\right)\left(ln\, t_i - \sum_{i=1}^{r} \alpha_i \, ln\, t_i\right)}{\sum_{i=1}^{r} \alpha_i \left(ln\, t_i - \sum_{i=1}^{r} \alpha_i \, ln\, t_i\right)^2} \tag{6}$$

$$\widehat{\lambda} = exp\left(\sum_{i=1}^{r} \alpha_i \, ln\, t_i - \frac{1}{\widehat{\alpha}} \sum_{i=1}^{r} \alpha_i \, ln\left(-ln f_n(t_i)\right)\right) \tag{7}$$

where α_i and $f_n(t_i)$ are as follows, respectively:

Table 1
Continuous interval-censored data table

Intervals	Frequency
$(0,\ t_1]$	n_1
$(t_1,\ t_2]$	n_2
$(t_2,\ t_3]$	n_3
...	...
$(t_{r-1},\ t_r]$	n_r
$(t_r,\ +\infty)$	$n - \sum_{i=1}^{r} n_i$

$$W_i = \left(1-\tfrac{R_i}{n}\right)^2\left(ln\left(1-\tfrac{R_i}{n}\right)\right)^2 \tag{8}$$

$$\alpha_i = W_i/\sum_{i=1}^{r} W_i \tag{9}$$

$$f_n(t_i) = 1-\tfrac{R_i}{n} \tag{10}$$

Proof. Reformulate the problem as follows:

$$Q = \sum_{i=1}^{r}\left[e^{-\left(\frac{t_i}{\lambda}\right)^\alpha}-\left(1-\tfrac{R_i}{n}\right)\right]^2 = \sum_{i=1}^{r}\left[e^{-\left(\frac{t_i}{\lambda}\right)^\alpha}-e^{ln\left(1-\frac{R_i}{n}\right)}\right]^2 \tag{11}$$

Based on the concepts of the weighted least squares approach, we can use $e^{ln\left(1-\frac{R_i}{n}\right)}\left(-\left(\frac{t_i}{\lambda}\right)^\alpha - ln\left(1-\frac{R_i}{n}\right)\right)$ to approximate $e^{-\left(\frac{t_i}{\lambda}\right)^\alpha}-e^{ln\left(1-\frac{R_i}{n}\right)}$. Thus, the original formula simplifies to:

$$Q = \sum_{i=1}^{r}\left[e^{ln\left(1-\frac{R_i}{n}\right)}\left(-\left(\frac{t_i}{\lambda}\right)^\alpha - ln\left(1-\frac{R_i}{n}\right)\right)\right]^2$$

$$= \sum_{i=1}^{r}\left(1-\frac{R_i}{n}\right)^2\left(e^{ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)}-e^{ln\left(\frac{t_i}{\lambda}\right)^\alpha}\right)^2 \tag{12}$$

Employing the aforementioned technique once more to approximate $exp\left(ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\right)-exp\left(ln\left(\frac{t_i}{\lambda}\right)^\alpha\right)$ with $exp\left(ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\right) * \left(ln\left(\frac{t_i}{\lambda}\right)^\alpha - ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\right)$, the original expression is transformed into:

$$Q = \sum_{i=1}^{r}\left(1-\frac{R_i}{n}\right)^2\left(ln\left(1-\frac{R_i}{n}\right)\right)^2\left(ln\left(\frac{t_i}{\lambda}\right)^\alpha - ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\right)^2$$

$$= \sum_{i=1}^{r}\left(1-\frac{R_i}{n}\right)^2\left(ln\left(1-\frac{R_i}{n}\right)\right)^2\left(\alpha\, ln\frac{t_i}{\lambda} - ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\right)^2 \tag{13}$$

Let $W_i = \left(1-\frac{R_i}{n}\right)^2\left(ln\left(1-\frac{R_i}{n}\right)\right)^2$:

$$Q = \sum_{i=1}^{r} W_i\left(\alpha\, ln\frac{t_i}{\lambda} - ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\right)^2 \tag{14}$$

To minimize the Q value, we compute the partial derivatives with respect α:

$$\frac{\partial Q}{\partial \alpha} = \sum_{i=1}^{r} 2W_i\left(\alpha\, ln\frac{t_i}{\lambda} - ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\right)\left(ln\frac{t_i}{\lambda}\right) = 0 \tag{15}$$

This is equivalent to

$$\alpha\sum_{i=1}^{r} W_i\left(ln\frac{t_i}{\lambda}\right)^2 = \sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right) ln\frac{t_i}{\lambda} \tag{16}$$

Thus, we have:

$$\alpha = \sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right) ln\frac{t_i}{\lambda} \Big/ \sum_{i=1}^{r} W_i\left(ln\frac{t_i}{\lambda}\right)^2 \tag{17}$$

Take the partial derivatives with respect λ:

$$\frac{\partial Q}{\partial \lambda} = \sum_{i=1}^{r} 2W_i\left(\alpha\, ln\frac{t_i}{\lambda} - ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\right)\left(-\frac{\alpha}{\lambda}\right) = 0 \tag{18}$$

It yields:

$$\alpha\sum_{i=1}^{r} W_i\left(ln\, t_i - ln\,\lambda\right) = \sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right) \tag{19}$$

Bringing $ln\,\lambda$ to the left-hand side of the equation, it becomes:

$$ln\,\lambda = \alpha\sum_{i=1}^{r} W_i\, ln\, t_i - \sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right) \Big/ \alpha\sum_{i=1}^{r} W_i \tag{20}$$

Simplifying the Equation (17) where $ln\frac{t_i}{\lambda} = lnt_i - ln\lambda$, we obtain:

$$\alpha = \sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right) ln\, t_i$$

$$- ln\,\lambda\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right) \Big/ \sum_{i=1}^{r} W_i\left(ln\, t_i - ln\,\lambda\right)^2 \tag{21}$$

Substituting the above Equation (20) into Equation (17), we obtain:

$$\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right) ln\, t_i$$

$$-\frac{\alpha\sum_{i=1}^{r} W_i\, ln\, t_i - \sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)}{\alpha\sum_{i=1}^{r} W_i}\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)$$

$$= \alpha\sum_{i=1}^{r} W_i\left(ln\, t_i - \frac{\sum_{i=1}^{r} W_i\, ln\, t_i}{\sum_{i=1}^{r} W_i} + \frac{\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)}{\alpha\sum_{i=1}^{r} W_i}\right)^2 \tag{22}$$

Multiply both sides by α simultaneously:

$$\alpha\left(\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right) ln\, t_i\right.$$

$$-\frac{\sum_{i=1}^{r} W_i\, ln\, t_i}{\sum_{i=1}^{r} W_i}\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\Bigg)$$

$$+\frac{\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)}{\sum_{i=1}^{r} W_i}\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)$$

$$= \sum_{i=1}^{r} W_i\left(\left(ln\, t_i - \frac{\sum_{i=1}^{r} W_i\, ln\, t_i}{\sum_{i=1}^{r} W_i}\right)\alpha\right.$$

$$\left. + \frac{\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)}{\sum_{i=1}^{r} W_i}\right)^2 \tag{23}$$

Upon expansion and simplification, we can derive the following:

$$\alpha = \frac{\sum_{i=1}^{r} W_i\, ln\left(-ln\left(1-\frac{R_i}{n}\right)\right)\left(ln\, t_i - \frac{\sum_{i=1}^{r} W_i\, ln\, t_i}{\sum_{i=1}^{r} W_i}\right)}{\sum_{i=1}^{r} W_i\left(ln\, t_i - \frac{\sum_{i=1}^{r} W_i\, ln\, t_i}{\sum_{i=1}^{r} W_i}\right)^2} \tag{24}$$

Dividing both numerator and denominator of Equation (24) by $\sum_{i=1}^{r} W_i$, then substituting with Equation (9), we get:

$$\alpha = \frac{\sum_{i=1}^{r} \alpha_i \ln\left(-\ln\left(1-\frac{R_i}{n}\right)\right)\left(\ln t_i - \sum_{i=1}^{r} \alpha_i \ln t_i\right)}{\sum_{i=1}^{r} \alpha_i (\ln t_i - \sum_{i=1}^{r} \alpha_i \ln t_i)^2} \tag{25}$$

Further substitution using Equation (10), we obtain:

$$\alpha = \frac{\sum_{i=1}^{r} \alpha_i \ln(-\ln f_n(t_i))(\ln t_i - \sum_{i=1}^{r} \alpha_i \ln t_i)}{\sum_{i=1}^{r} \alpha_i (\ln t_i - \sum_{i=1}^{r} \alpha_i \ln t_i)^2} \tag{26}$$

Similarly, utilizing Equation (9) for substitution into Equation (20), we can obtain an explicit expression for λ.

$$\lambda = exp\left(\sum_{i=1}^{r} \alpha_i \ln t_i - \frac{1}{\alpha} \sum_{i=1}^{r} \alpha_i \ln(-\ln f_n(t_i))\right) \tag{27}$$

In practical applications, detection intervals are sometimes uniform, like in the street light failure example, where detection occurs monthly. Based on Proposition 1, if the intervals are equidistant, denoted as $(0, c], \cdots, ((r-1)c, rc]$, then the integral in Equations (6) and (7) can be simplified as follows:

$$\begin{aligned} &\ln ic - \sum_{i=1}^{r} \alpha_i \ln ic \\ &= \ln i + \ln c - \sum_{i=1}^{r} \alpha_i (\ln i + \ln c) \\ &= \ln i - \sum_{i=1}^{r} \alpha_i \ln i \end{aligned} \tag{28}$$

$$\widehat{\lambda} = exp\left(\sum_{i=1}^{r} \alpha_i \ln t_i\right) / exp\left(\frac{1}{\alpha} \sum_{i=1}^{r} \alpha_i \ln(-\ln f_n(t_i))\right) \tag{29}$$

It can be observed that the simplified expression in (28) does not depend on c, meaning the parameter estimate $\widehat{\alpha}$ is also independent of the interval. Moreover, the denominator of parameter $\widehat{\lambda}$ is not influenced by c. Hence, we only need to concentrate on the numerator.

$$\begin{aligned} &exp\left(\sum_{i=1}^{r} \alpha_i \ln t_i\right) \\ &= exp\left(\sum_{i=1}^{r} \alpha_i \ln i + \sum_{i=1}^{r} \alpha_i \ln c\right) \\ &= c * exp(\textstyle\sum_{i=1}^{r} \alpha_i \ln i) \end{aligned} \tag{30}$$

Thus, we find that the scale parameter has a linear relationship with the interval. Let's illustrate this with a simple example. Consider two equidistant test intervals shown in the Table 2, each with the same frequency.

When we substitute the data from the table into Equations (6) and (7), we obtain the shape and scale parameters for Equal Interval 1 as 1.8793 and 4.3102, respectively. For Equal Interval 2, the shape and scale parameters are 1.8793 and 8.6204, respectively. This means that the shape parameter estimated for both equal interval test intervals is the same, while the scale parameter for Equal Interval 2 is twice that of Equal Interval 1. Similarly, the ratio of their interval sizes is also 2. This pattern is insightful: firstly, the shape parameter is independent of the interval size c, indicating that the shape of the frequency distribution histogram directly determines the shape parameter, independent of the scale parameter. Secondly, the proportional relationship between the λ and the interval size c implies that as the interval c increases, λ increases as well. This aligns with the role of λ in determining the width of the Weibull distribution.

Table 2
Interval-censored data with two equidistant intervals

Equal interval1	Frequency	Equal interval2	Frequency
(0, 1]	4	(0, 2]	4
(1, 2]	7	(2, 4]	7
(2, 3]	8	(4, 6]	8
(3, 4]	9	(6, 8]	9
(4, 5]	10	(8, 10]	10

4. Parameter Estimation Simulation and Discussion

4.1. Simulation test

To demonstrate the effectiveness of the analytical solution for parameter estimation, we generate random numbers for Weibull distributions with three typical shape parameters, respectively, and conduct random simulations.

Firstly, we employ the Monte Carlo method to construct continuous interval deletion data akin to Table 3. Subsequently, utilizing the parameter estimation model established by Proposition 1, we iteratively solve it through 1000 repetitions of experiments. From this, we derive the mean, variance, and mean squared error (MSE) of the shape and scale parameters, thus validating the feasibility and efficacy of the aforementioned approach.

Here, based on the Weibull distribution density function curves (Figure 2), we design different continuous interval deletion data sets based on the characteristics of the Weibull distribution under three sets of parameters. We perform three sets of experiments, with each group's samples adhering to the Weibull distributions Weibull (0.8, 3), Weibull (2.5, 6), and Weibull (10, 12). These distributions represent distinct failure patterns for products: the increasing failure rate, the constant failure rate, and the decreasing failure rate types. To ensure non-empty data within each interval, we select intervals such that each interval lies within a significantly nonzero segment of the density function. We make the following reasonable divisions: for Weibull (0. 8, 3), the intervals are set as (0, 3], (3, 5], (5, 7], (7, 10]; for Weibull (2.5, 6), the intervals are set as (0, 3], (3, 6], (6, 8], (8, 11]; and for Weibull (10, 12), the intervals are set as (0, 9], (9, 10], (10, 11], (11, 12], as shown in Table 3. This design is rational, as in practical engineering, we often utilize prior statistical information to plan the next steps of experimentation, thereby avoiding unnecessary expenditure of experimental resources, time, and labor costs.

From Table 4, we observe that as the amount of interval data increases, the parameter estimation error obtained using this method gradually decreases. The estimation relative error here is calculated using the formula $\frac{|\widehat{\theta}-\theta|}{\theta}$, $\theta = \alpha,\ \beta$. In Case 1, with a sample size of 30, the shape parameter estimation error is 0.925%, and the scale parameter estimation error is 2.27%. In Case 2, with a sample size of 100, the shape parameter estimation error is 1.144%, and the scale parameter estimation error is 0.91%. In Case 3, with a sample size of 100, the shape parameter estimation

Table 3
Interval-censored data example

Case	1	2	3
Real	Weibull	Weibull	Weibull
Parameter	(0.8, 3)	(2.5, 6)	(10, 12)
Interval1	(0, 3]	(0, 3]	(0, 9]
Interval2	(3, 5]	(3, 6]	(9, 10]
Interval3	(5, 7]	(6, 8]	(10, 11]
Interval4	(7, 10]	(8, 11]	(11, 12]

Figure 2
Weibull distribution density function curves under three different cases

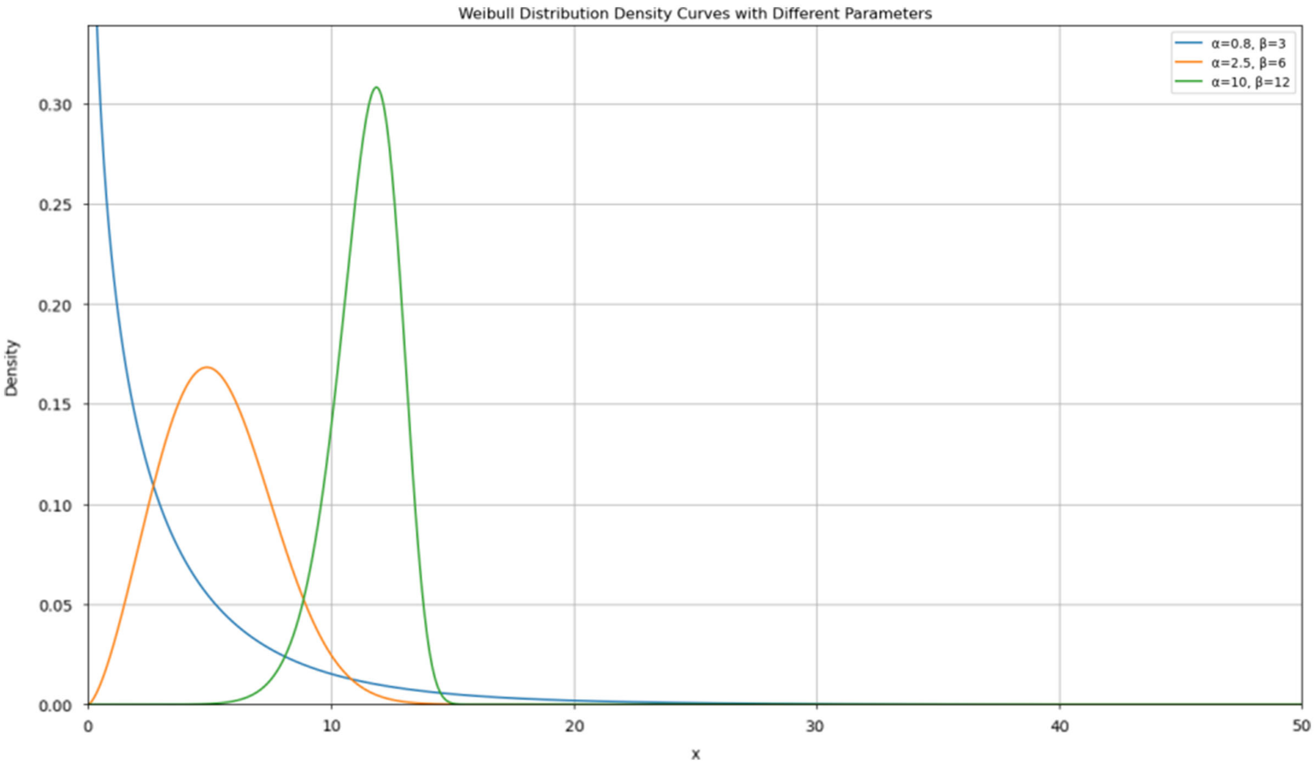


Table 4
Interval-censored data example

Case	Sample size	$\widehat{\alpha}$	$\widehat{\lambda}$	MSE($\widehat{\alpha}$)	MSE($\widehat{\lambda}$)
1	20	0.7681	3.1531	0.0697	1.4653
	30	0.7926	3.0681	0.0537	0.8781
	50	0.7939	2.9929	0.0322	0.5049
	100	0.8061	2.9947	0.0171	0.2636
	500	0.7993	3.0066	0.0030	0.0496
	1000	0.8008	3.0003	0.0015	0.0236
2	20	2.1657	6.2741	0.2466	0.4612
	30	2.3061	6.1544	0.1421	0.2544
	50	2.3922	6.0760	0.1007	0.1324
	100	2.4714	6.0544	0.0590	0.0836
	500	2.5016	6.0004	0.0129	0.0148
	1000	2.5048	6.0018	0.0064	0.0073
3	20	9.1386	12.0489	5.7461	0.2014
	30	9.4702	12.0070	4.4594	0.0974
	50	9.8842	12.0075	3.0783	0.0567
	100	9.9154	12.0105	1.4756	0.0270
	500	9.9912	11.9970	0.3118	0.0048
	1000	10.0057	12.0006	0.1517	0.0025

error is 0.846%, and the scale parameter estimation error is 0.0875%. It can be seen that the estimation effect is ideal. When the sample size is small, the larger deviation is partly due to insufficient sample information, which is a normal phenomenon.

Figures 3–8 allow us to visually see that the estimation accuracy significantly increases with the increase in sample size, further demonstrating the effectiveness of this method.

4.2. Method comparison

We utilize the interval-censored data from Tan [24], with specific information presented in Table 5. The data consist of 157 crack data points from identical components, divided into 9 intervals, with 5 components in one interval, and the rest are similar.

Similarly, as can be seen from Table 6, the proposed method significantly outperforms the results presented in Tan [24] and Nelson [25] in terms of fitting the original interval data. The sum of squared residuals is the value obtained when the estimated parameter values are substituted into Q. It reflects the fitting error between the Weibull probability density curve and the frequency histogram of the interval data.

For time interval testing experiments, we compare our proposed method with existing methods. Since the specific failure times or lifetimes of the products during each time interval are not known, we typically use linear interpolation to estimate these data, with the notation being the same as shown in the aforementioned table.

Let t_{ij} denote the failure time of product in the-th time interval, its calculation method is as follows:

$$t_{ij} = t_{i-1} + \frac{j}{n_i + 1}(t_i - t_{i-1}),\ j = 1, \cdots, n_i \tag{31}$$

Subsequently, we utilize the approaches discussed in the second section, such as MLE and the method of moments, to estimate the parameters. Building upon the research of predecessors, we understand that MLE tends to perform better compared to the BLUE, least squares estimation, and method of moments [26–28]. Therefore, below we will primarily compare with the MLE, the MLE technique used is detailed in Joarder et al. [29].

Figure 3
Frequency distribution histogram of $\widehat{\alpha}$ in case 1

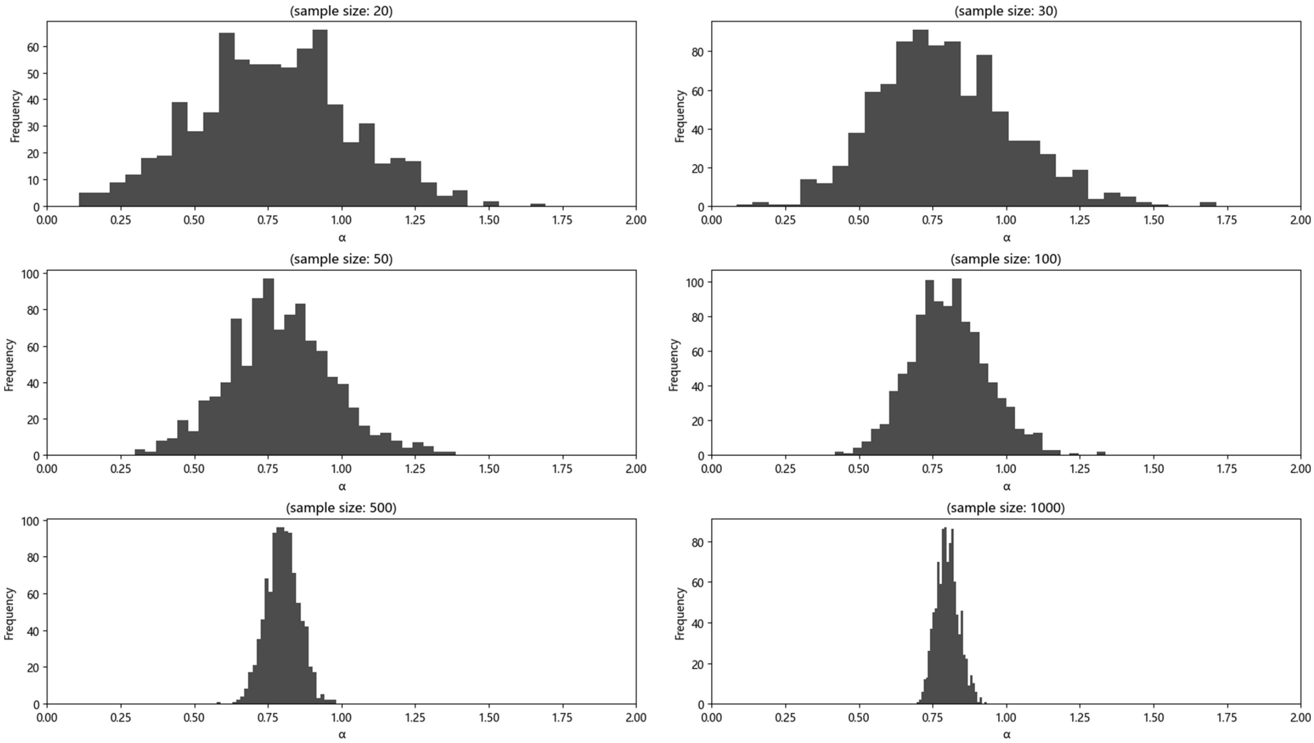


Figure 4
Frequency distribution histogram of $\widehat{\lambda}$ in case 1

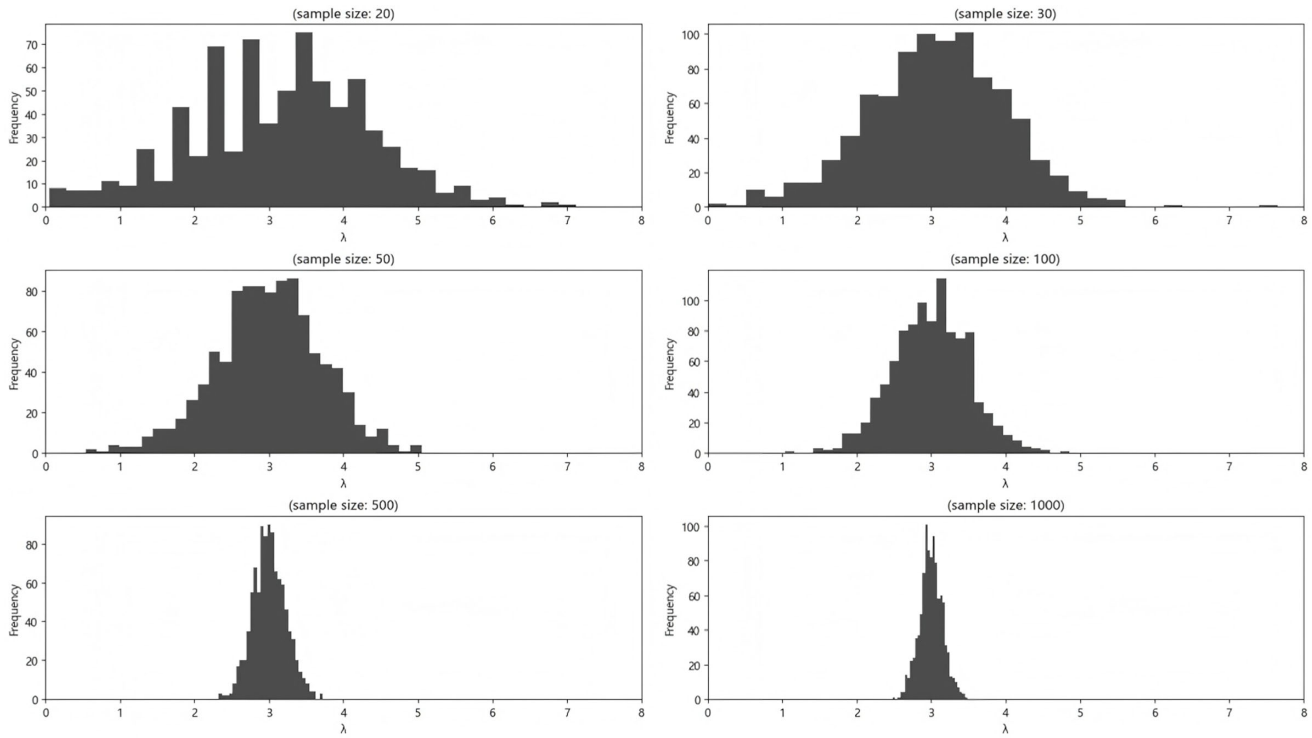


Figure 5
Frequency distribution histogram of $\widehat{\alpha}$ in case 2

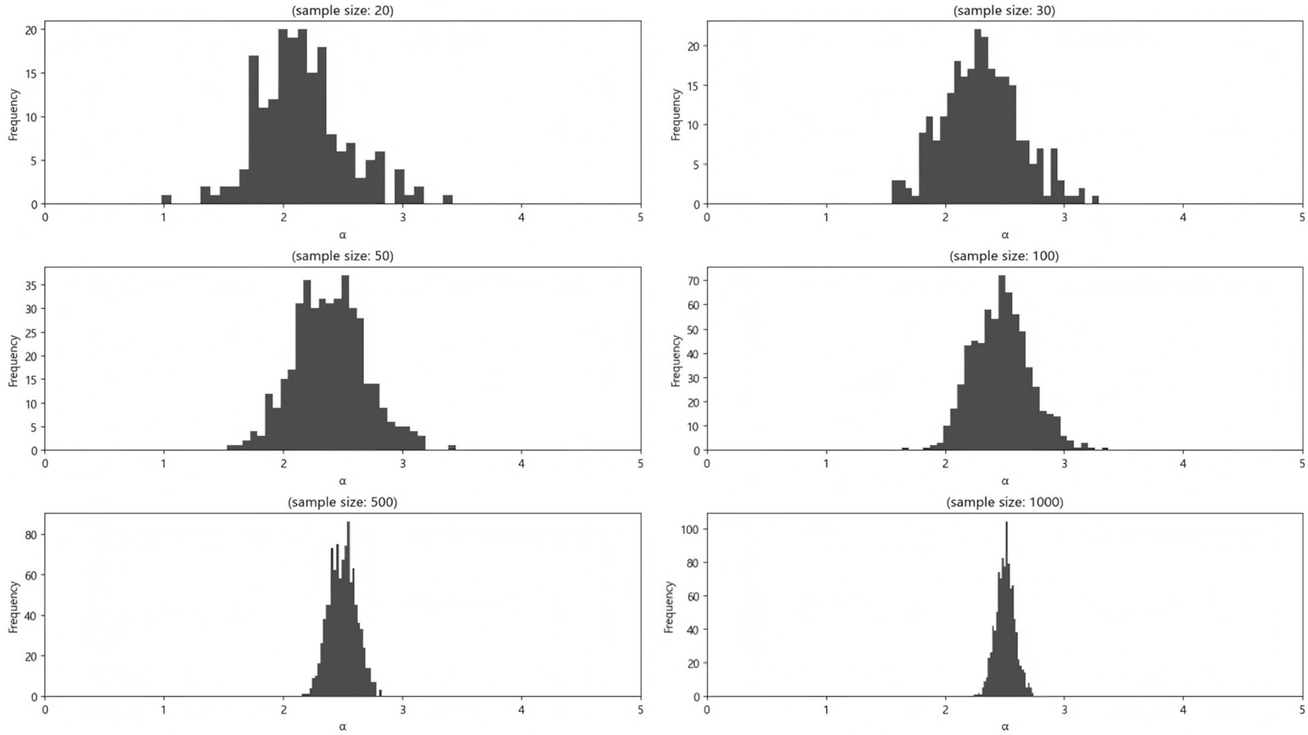


Figure 6
Frequency distribution histogram of $\widehat{\lambda}$ in case 2

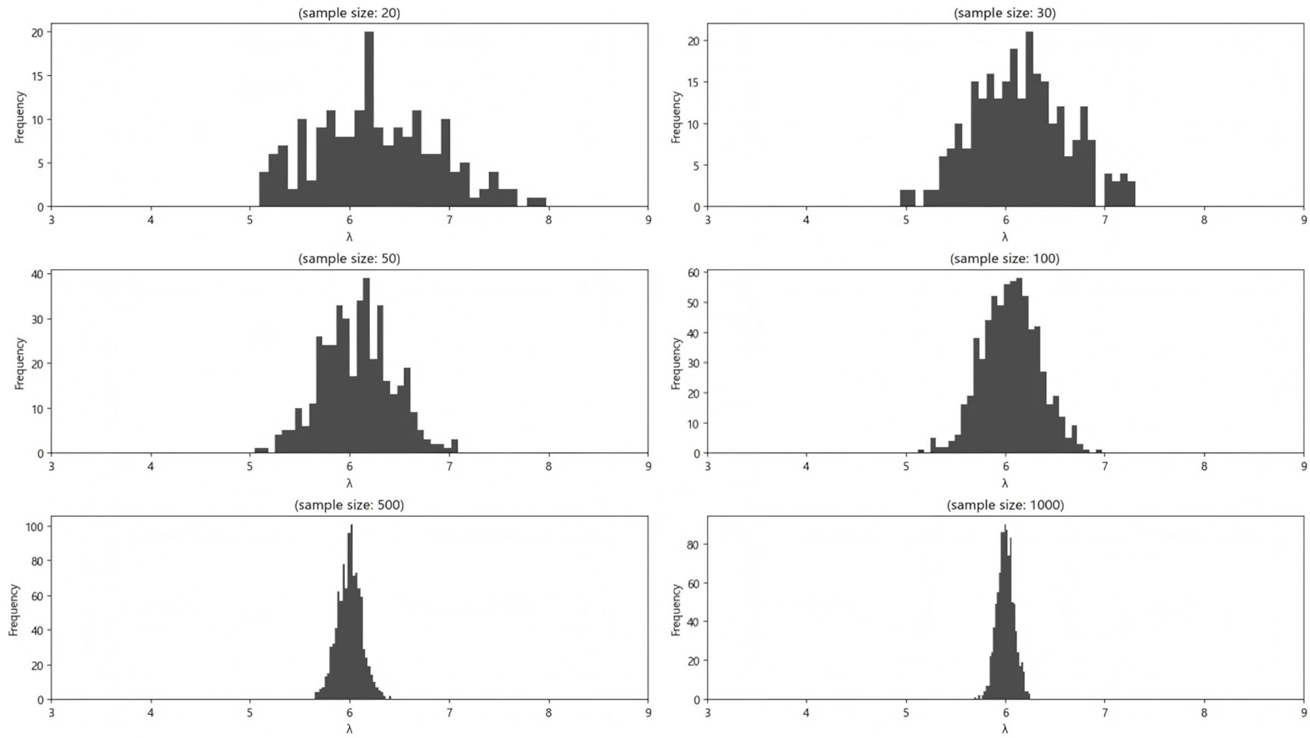


Figure 7
Frequency distribution histogram of $\widehat{\alpha}$ in case 3

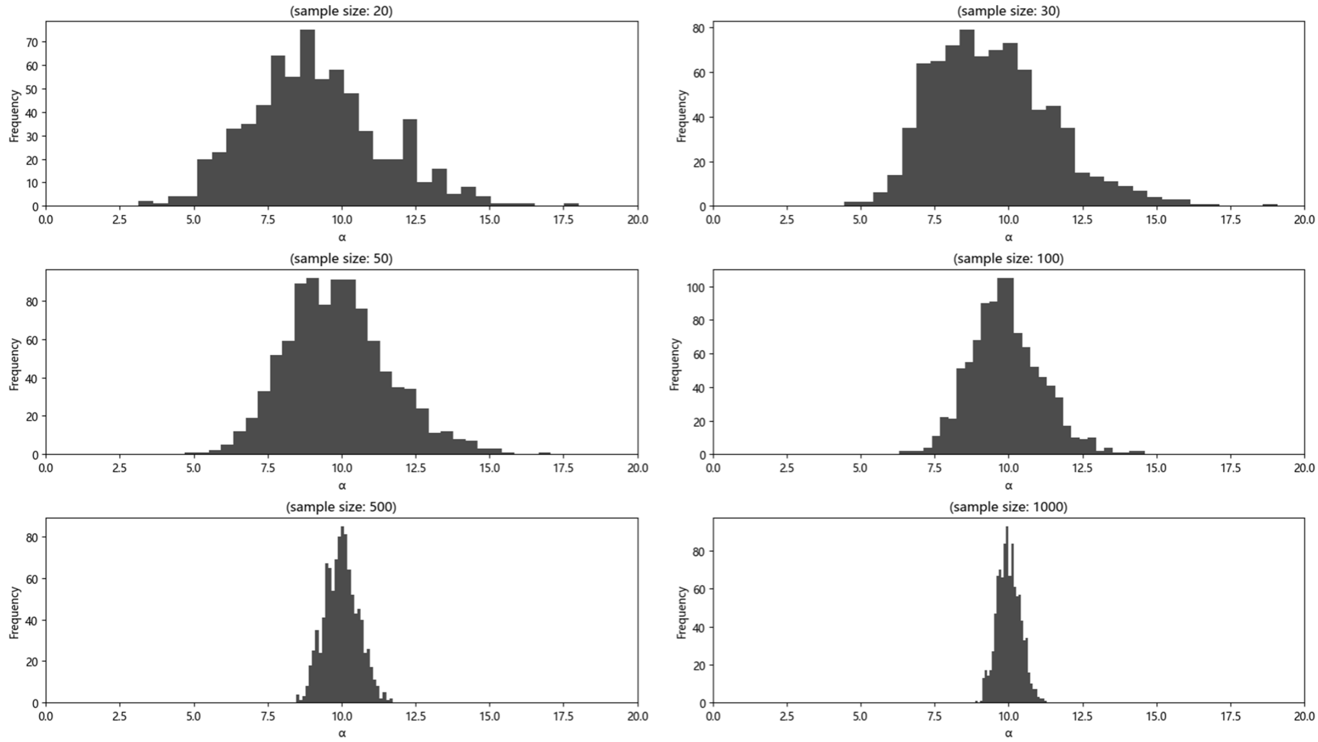


Figure 8
Frequency distribution histogram of $\widehat{\lambda}$ in case 3

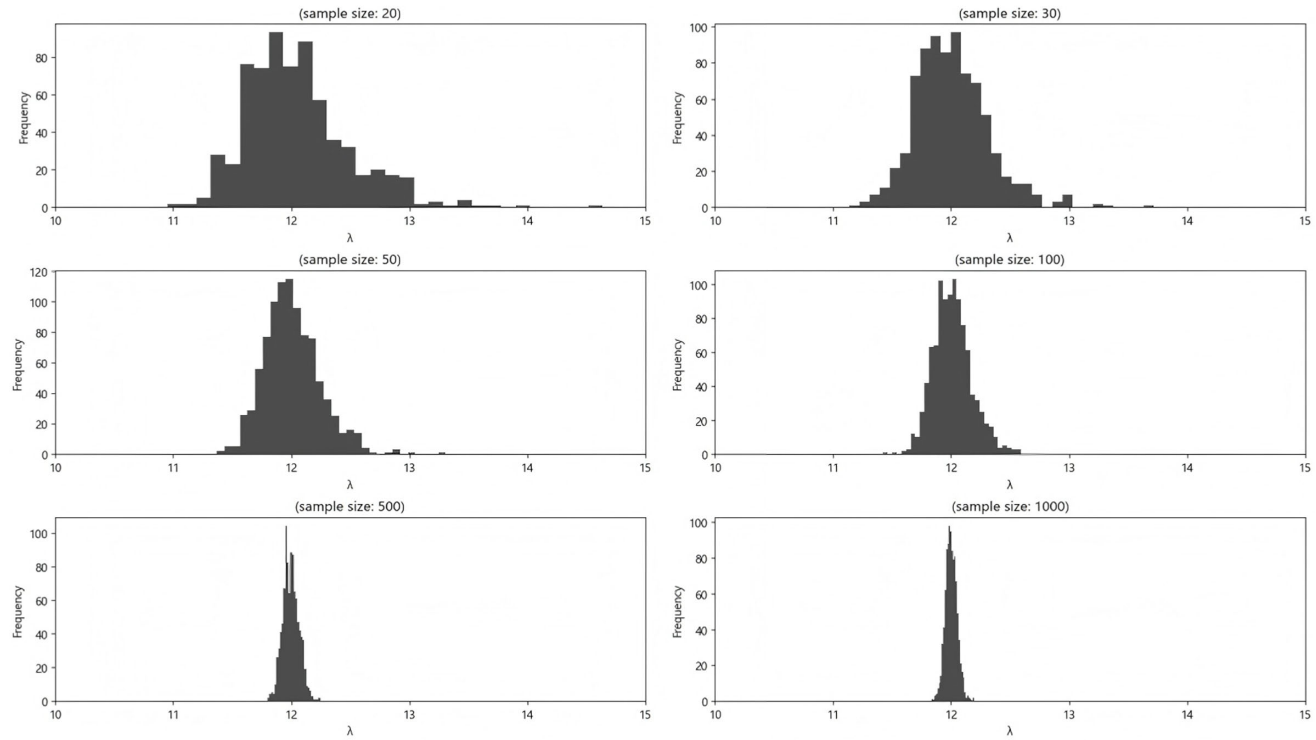


Table 5
Cracking data of 157 identical components

Intervals	Number in interval
(0, 6.12)	5
(6.12, 19.92)	6
(19.92, 29.64)	12
(29.64, 35.4)	18
(35.4, 39.72)	18
(39.72, 45.24)	2
(45.24, 52.32)	6
(52.32, 63.48)	17
(63.48, $+\infty$)	73

Table 6
One-way ANOVA results based on teaching subjects

Method	$\widehat{\alpha}$	$\widehat{\lambda}$	SSE
PM	1.5771	73.2497	**0.3817**
Results [24]	1.485	71.690	1.2812
Results [25]	1.486	71.687	1.2748

Table 7
Parameter estimation mean, variance, and standard error for PM and MLE in case 2 (Weibull(2.5, 6)) with a sample size of 100

Method	Proposed method	MLE
$\widehat{\alpha}$	2.4714	1.6274
$\widehat{\lambda}$	6.0544	4.8233
Var($\widehat{\alpha}$)	0.0583	0.0310
MSE($\widehat{\alpha}$)	0.0590	0.7924
Var($\widehat{\lambda}$)	0.0807	0.0921
MSE($\widehat{\lambda}$)	0.0836	1.4766

From Table 7, it is evident that there is a noticeable improvement with proposed method. The performance of MLE using linear interpolation may be compromised due to two factors: the interpolated data not following a Weibull distribution and the right-censored data.

4.3. Parameter estimation extension

For n test samples, with the lifespans of r products known and containing n-r right-censored data, to apply Proposition 1, consider the following two grouping methods. The failure time $t_0 = 0 < t_1 < \ldots < t_r$ is processed as follows:

$$\tau_i = \frac{t_i + t_{i-1}}{2},\ i = 1, 2, \cdots, r \tag{32}$$

$$\tau_i = t_i,\ i = 1, 2, \cdots, r \tag{33}$$

This will yield an interval for $(0,\ \tau_1]$, $(\tau_1,\ \tau_2]$, $\cdots$, $(\tau_{r-1},\ \tau_r]$ similar to the discussion above, and here $n_i = 1$, $i = 1, 2, \cdots, r$, so the parameter estimation can be the same as the previous formula.

We have designed three sets of experiments, in which we have agreed upon such symbolic representations for the conducted experiments, $\mathcal{K}(\alpha,\ \lambda,\ x,\ y)$, where α, λ denote the shape and scale parameters of the Weibull distribution, respectively, and x, y represent the percentage of censored data samples and the total sample size, respectively. The three sets of experiments designed here are as follows, $\mathcal{K}(0.8,\ 3,\ 80\%,\ 100)$, $\mathcal{K}(3,\ 6,\ 80\%,\ 100)$, $\mathcal{K}(10,\ 12,\ 80\%,\ 100)$, with the default number of repeated trials set to 1000. Similarly, we compare our proposed extended methods with the MLE, and the detailed comparison can be seen in the table below, where the method using Equation (32) is referred to as Proposed Method-1, and the method using Equation (33) is referred to as Proposed Method-2.

Table 8
Proposed method-1, proposed method-2, and MLE in the application comparison of $\mathcal{K}$ (0.8, 3, 80%, 100)

Method	$\widehat{\alpha}$	$\widehat{\lambda}$	RE($\widehat{\alpha}$)	RE($\widehat{\lambda}$)
Proposed Method-1	**0.8031**	3.0258	**0.3875%**	0.8600%
Proposed Method-2	0.7931	**3.0040**	0.8625%	**0.1333%**
MLE	0.8135	3.0300	1.6875%	1.0000%

Table 9
Proposed method-1, proposed method-2, and MLE in the application comparison of $\mathcal{K}$ (3, 6, 80%, 100)

Method	$\widehat{\alpha}$	$\widehat{\lambda}$	RE($\widehat{\alpha}$)	RE($\widehat{\lambda}$)
Proposed Method-1	**2.9888**	**6.0175**	**0.3733%**	**0.2917%**
Proposed Method-2	2.9709	5.9816	0.9700%	0.3067%
MLE	3.0529	5.9810	1.7633%	0.3167%

Table 10
Proposed method-1, proposed method-2, and MLE in the application comparison of $\mathcal{K}$ (10, 12, 80%, 100)

Method	$\widehat{\alpha}$	$\widehat{\lambda}$	RE($\widehat{\alpha}$)	RE($\widehat{\lambda}$)
Proposed Method-1	**10.0041**	**11.9932**	**0.0410%**	**0.0567%**
Proposed Method-2	9.9011	11.9898	0.9890%	0.0850%
MLE	10.1768	11.9871	1.7680%	0.1075%

From Tables 8–10, it is evident that when retaining only 80% of the known observed data, the extended method using Equation (32) exhibits estimation relative errors (RE) for the shape parameter of 0.3875%, 0.3733%, and 0.0410% in the three trials, respectively. In contrast, the method using Equation (33) shows estimation errors of 0.8625%, 0.9700%, and 0.9890% for the shape parameter across the same trials. These errors are significantly lower than the relative errors of the MLE method, which are 1.6875%, 1.7633%, and 1.7680%, respectively. This affirms the effectiveness of the extended method in fixed-failure-rate censoring tests. Additionally, the approach of converting points to intervals using Equation (32) performs better than that using Equation (33). That is, utilizing the midpoints between the original sample values as interval endpoints can make better use of the original sample information, thereby yielding more precise parameter estimates. Simultaneously, we can observe that the method we proposed yields a MSE slightly larger than that of the MLE when estimating parameters. This is related to the fact that we only use interval values without specific numerical data. However, their standard errors (SdE) are quite close, and the estimation is closer to being unbiased, so we can still consider it to be slightly superior to MLE.

Table 11
Constant failure rate sequential censoring data explanation table

Intervals	"failure" or "damage" count	The number of non-failed products removed
$(0,\ t_1]$	n_1	k_1
$(t_1,\ t_2]$	n_2	k_2
$(t_2,\ t_3]$	n_3	k_3
$\cdots$	$\cdots$	$\cdots$
$(t_{r-2},\ t_{r-1}]$	n_{r-1}	k_{r-1}
$(t_{r-1},\ t_r]$	n_r	k_r

In practical applications, it is also common to encounter the situation where test samples are removed, that is, continuous sequential interval-censored data. For this scenario, with a slight modification of Proposition 1, Corollary 2 can be derived. The distinction between continuous sequential interval-censored data and continuous interval-censored data is that a portion of the samples that have not yet failed is removed at each inspection interval. Specifically, n products are subjected to a life test simultaneously. Within the interval $(0,\ t_1]$, n_1 products fail and k_1 non-failed products are removed. Within the interval $(t_1,\ t_2]$, n_2 products fail and k_2 non-failed products are removed, and this process continues sequentially. By the time we reach the interval $(t_{r-1},\ t_r]$, a total of products have failed and $k_r = n - \sum_{i=1}^{r} n_i - \sum_{i=1}^{r-1} k_i$ non-failed products have been removed. This can be represented more clearly and intuitively in Table 11 as follows.

In the original derivation of Proposition 1, we needed to make $exp\left(-\left(\frac{t_i}{\lambda}\right)^{\alpha}\right)$ and $1-\frac{\sum_{j=1}^{i} n_j}{n}$ as close as possible. Here, we need to make $exp\left(-\left(\frac{t_i}{\lambda}\right)^{\alpha}\right)$ and $1-\frac{\sum_{j=1}^{i} n_j}{n-\sum_{j=1}^{i} k_j}$ as close as possible, which is to minimize the following expression:

$$M = \sum_{i=1}^{r}\left[e^{-\left(\frac{t_i}{\lambda}\right)^{\alpha}} - \left(1-\frac{R_i}{n-\sum_{j=1}^{i} k_j}\right)\right]^2 \tag{34}$$

Corollary 2. Given that $\mathrm{R_i} = \sum_{j=1}^{i} \mathrm{n_j}$, the solution that minimizes expression (34) is as follows:

$$\widehat{\alpha} = \frac{\sum_{i=1}^{r} \beta_i\, ln(-ln\, g_n(t_i))(ln\, t_i - \sum_{i=1}^{r} \beta_i\, ln\, t_i)}{\sum_{i=1}^{r} \beta_i\,(ln\, t_i - \sum_{i=1}^{r} \beta_i\, ln\, t_i)^2} \tag{35}$$

$$\widehat{\lambda} = exp\left(\sum_{i=1}^{r} \beta_i\, ln\, t_i - \frac{1}{\widehat{\alpha}}\sum_{i=1}^{r} \beta_i\, ln\,(-ln\, g_n(t_i))\right) \tag{36}$$

β_i and $g_n(t_i)$ are respectively:

$$F_i = \left(1-\frac{R_i}{n-\sum_{j=1}^{i} k_j}\right)^2 \left(ln\left(1-\frac{R_i}{n-\sum_{j=1}^{i} k_j}\right)\right)^2 \tag{37}$$

$$F_i / \sum_{i=1}^{r} F_i \triangleq \beta_i \tag{38}$$

$$1-\frac{R_i}{n-\sum_{j=1}^{i} k_j} \triangleq g_n(t_i) \tag{39}$$

The proof here is the same as for Proposition 1.

For the parameter estimation of the three-parameter Weibull distribution, we can provide a rough estimate for the location parameter γ using the method from Guo et al. [30], refer to Equation (40) for the specific form. For the shape and scale parameters, we can use formula (6) and (7), respectively.

$$\gamma = min(t_i) - \frac{1}{n} \tag{40}$$

We summarize the proposed method as follows: The analytical formula driven by Proposition 1 is not only applicable to the original continuous interval-censored data but also to fixed-censoring data and continuous sequential interval-censored data. By combining the methods used for the latter two, it can be extended to fixed-censoring sequential tests. First, use Equation (33) to transform it into continuous sequential interval-censored data and then apply the corresponding method; regarding the parameter estimation of the three-parameter Weibull distribution, it should be noted that we can only obtain a rough estimate. After obtaining the location parameter from Equation (40), the original data can be combined with the location parameter estimate to reduce it to a two-parameter case, thereby obtaining further estimates. For the case of the three-parameter model, although the aforementioned steps can simplify it to a two-parameter model for solving, there is no significant advantage compared to the numerical algorithms in the literature. The issue of the analytical solution for the three-parameter model still requires further research. The detailed flowcharts for various methods can be seen in Figure 9.

The advantages of this method are reflected in the following aspects: First, its analytical simplicity is achieved by providing closed-form analytical solutions (Equations 6 and 7), eliminating the iterative optimization processes required by traditional numerical algorithms (e.g., EM or MCMC). Second, it significantly improves computational efficiency by greatly reducing computational overhead. Additionally, the method demonstrates broader applicability, extending to fixed-censoring tests, sequential interval-censored tests, and being partially applicable to parameter estimation for three-parameter Weibull distributions. Finally, its superior accuracy is validated through Monte Carlo simulations and comparisons with existing methods in the literature, such as outperforming MLE with linear interpolation in terms of relative error and fitting residuals (see Tables 6–8). These strengths provide robust theoretical support for practical applications in engineering and medical fields.

The method also has several limitations. First, it is restricted to the two-parameter Weibull distribution, as the three-parameter case requires additional approximations and lacks a robust analytical solution. Second, its performance is heavily dependent on the Weibull distribution assumption; significant deviations of the data from this model may lead to suboptimal accuracy. Finally, the method exhibits sensitivity to interval design, particularly when intervals are unequally spaced, which could introduce biases in parameter estimation and affect reliability in practical applications. These constraints highlight the need for careful model validation and interval partitioning when implementing the approach.

5. Fitting Application of Hong Kong Casualty Data

The data from the World Health Organization are segmented into age groups: 0, 1–4, 5–14, 15–24, 25–34, 35–54, 55–74, and 75+,

Figure 9
Analytical method application flowchart for different types of data

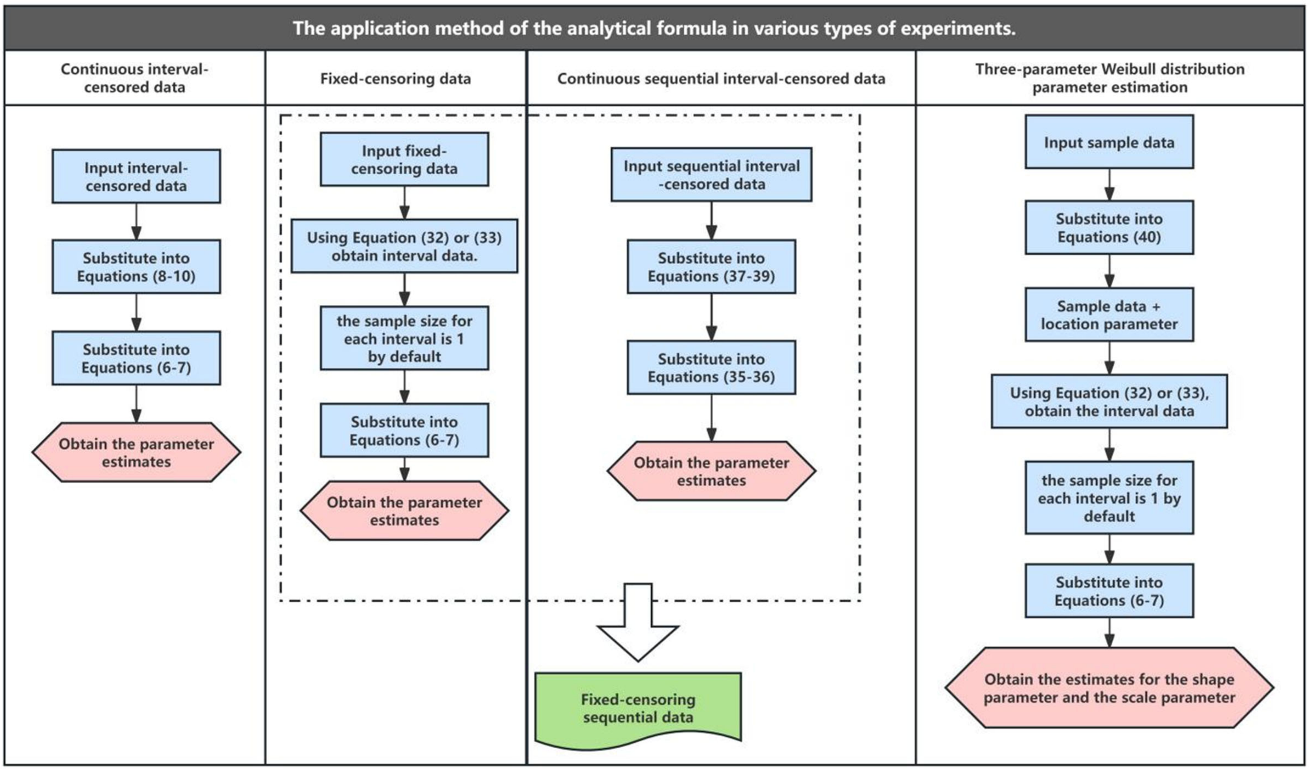
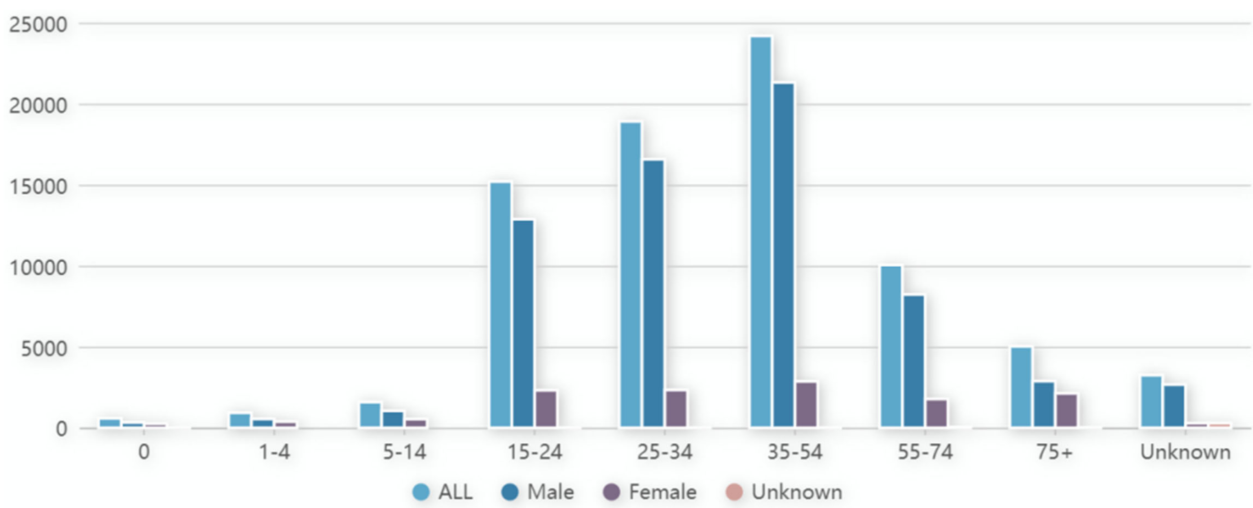


Figure 10
Table of injury-related deaths by age group in Hong Kong, China, for the year 2022



with each age group featuring three attributes: male, female, and unknown, as illustrated in Figures 10.

Using the derived explicit expression for parameter estimation, the shape parameter and scale parameter are 2.0279 and 42.8871, respectively, with the corresponding probability density curve shown in Figure 11. The reliability function curve shows the median life and characteristic life, as shown in Figure 12. As shown in Figure 13, the probability density function of the Weibull distribution provides a good fit to the histogram of casualties in Hong Kong.

Based on the obtained Weibull distribution parameter estimates, we are able to make predictions for the unknown mortality data. The specific predicted values are provided in Table 12.

Figure 11
Probability density curve corresponding to parameter estimates (2.0279 and 42.8871)

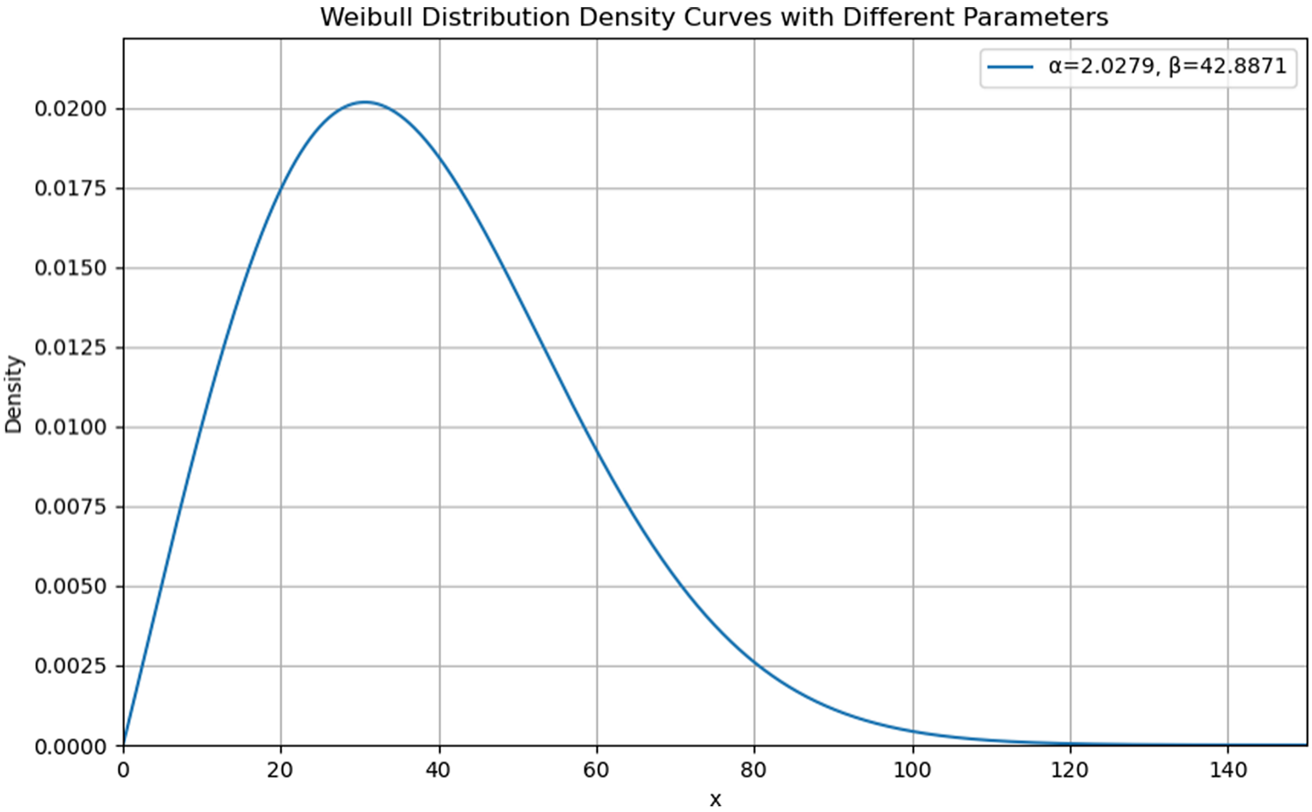


Figure 12
Characteristic life and median life corresponding to parameter estimates (2.0279 and 42.8871)

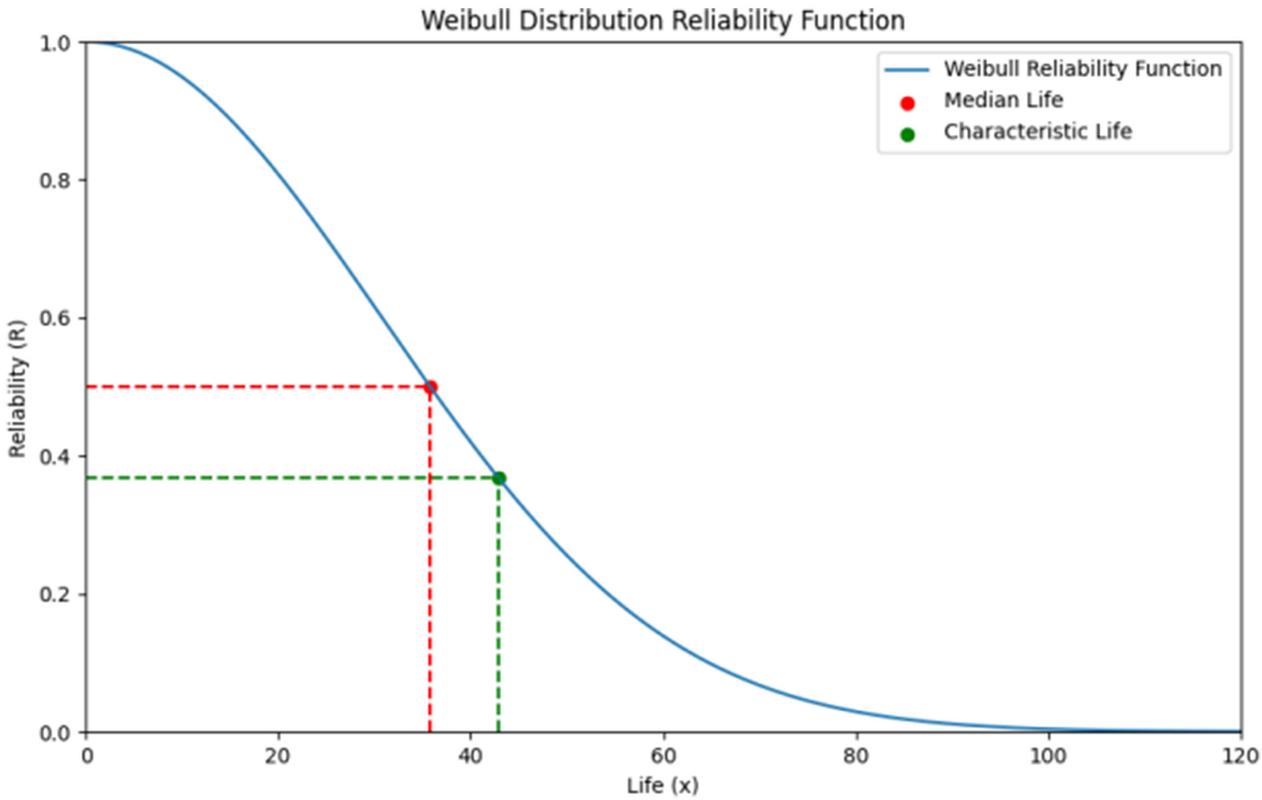


Figure 13
Probability density curve and data frequency distribution histogram fitting effect chart

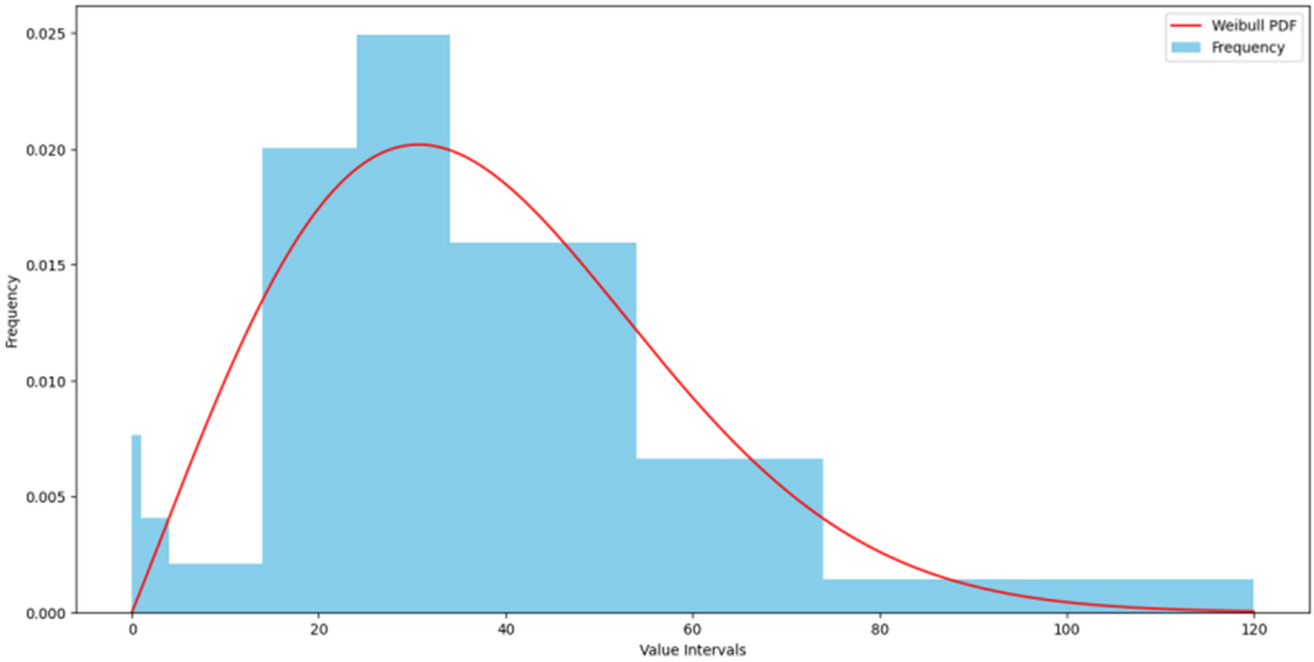


Table 12
Constant failure rate sequential censoring data explanation table

Age group	Probability value	Population forecast
0	0.0005	2
1–4	0.0076	25
5–14	0.0900	293
15–24	0.1671	543
25–34	0.1993	648
35–54	0.3328	1082
55–74	0.1541	501
75–120	0.0483	156

6. Conclusions

Against the backdrop of the three major challenges faced by Weibull distribution parameter estimation under censored data, this paper, grounded in extreme value statistics theory, innovatively derives a closed-form solution for parameter estimation by constructing a bilinear optimization model within a weighted least squares framework. Theoretical analysis and Monte Carlo simulations demonstrate that the proposed method achieves a computational complexity of *O(n)* while reducing the relative error of parameter estimation under interval-censored data to below 1%. Moreover, the closed-form solution provides a rigorous mathematical foundation for online lifetime prediction in reliability engineering. This advancement not only resolves the precision-efficiency trade-off inherent in traditional methods but also opens new methodological pathways for survival analysis under complex censoring mechanisms. Furthermore, the method is extendable to time-censored and failure-censored tests, as well as three-parameter Weibull distribution scenarios, as detailed in Figure 9. The explicit solution we proposed has also been applied in the World Health Organization's study of casualty data. This solution not only successfully fits the casualty distribution but also provides effective predictions for unknown casualty data. So it is recommended to apply this method for parameter estimation of interval-censored data.

Funding Support

Supports from the interdisciplinary research project of Central financial special funds (No.JGK05241001) are greatly acknowledged.

Ethical Statement

This study does not contain any studies with human or animal subjects performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

Data Availability Statement

The data that support the findings of this study are openly available in World Health Organization at https://platform.who.int/mortality/themes/theme-details/mdb/injuries.

Author Contribution Statement

Yang Yu: Conceptualization, Methodology, Software, Validation, Formal analysis, Data curation, Writing – original draft, Writing – review & editing, Visualization. **Jianguo Gong:** Resources, Funding acquisition. **Kunping Zhu:** Conceptualization, Methodology, Investigation, Supervision, Project administration.

References

[1] Tversky, A., & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases: Biases in judgments reveal some heuristics of thinking under uncertainty. *Science*, *185*(4157), 1124–1131.

[2] Gaidai, O., Li, H., Cao, Y., Liu, Z., Zhu, Y., & Sheng, J. (2024). Wind turbine gearbox reliability verification by multivariate Gaidai reliability method. *Results in Engineering*, *23*, 102689.

[3] Jiang, D., Han, Y., Cui, W., Wan, F., Yu, T., & Song, B. (2023). An improved modified Weibull distribution applied to predict the reliability evolution of an aircraft lock mechanism. *Probabilistic Engineering Mechanics*, *72*, 103449.

[4] Fan, L., Hu, Z., Ling, Q., Li, H., Qi, H., & Chen, H. (2023). Reliability analysis of computed tomography equipment using the q-Weibull distribution. *Engineering Reports*, *5*(7), e12613.

[5] Das, D., Samanta, G. C., Barman, A., De, P. K., & Mohanta, K. K. (2022). A recovery mathematical model for the impact of supply chain interruptions during the lockdown in COVID-19 using two warehouse perishable inventory policies. *Results in Control and Optimization*, *9*, 100184.

[6] Das, D., & Samanta, G. C. (2023). An EOQ model for a two warehouse system with price and time dependent demand and instantaneous deterioration during COVID-19 pandemic. *Research Square* [Preprint]. Available from: https://doi.org/10.21203/rs.3.rs-2777984/v1

[7] Barman, A., Chakraborty, A. K., Goswami, A., Banerjee, P., & De, P. K. (2023). Pricing and inventory decision in a two-layer supply chain under the weibull distribution product deterioration: An application of NSGA-II. *RAIRO-Operations Research*, *57*(4), 2279–2300.

[8] Örkcü, H. H., Özsoy, V. S., Aksoy, E., & Dogan, M. I. (2015). Estimating the parameters of 3-p Weibull distribution using particle swarm optimization: A comprehensive experimental comparison. *Applied Mathematics and Computation*, *268*, 201–226.

[9] Kundu, D., & Dey, A. K. (2009). Estimating the parameters of the Marshall-Olkin bivariate Weibull distribution by EM algorithm. *Computational Statistics & Data Analysis*, *53*(4), 956–965.

[10] Jiang, H., Xie, M., & Tang, L. C. (2008). Markov chain Monte Carlo methods for parameter estimation of the modified Weibull distribution. *Journal of Applied Statistics*, *35*(6), 647–658.

[11] Lai, C. D., Murthy, D. N. P., & Xie, M. (2011). Weibull distributions. *Wiley Interdisciplinary Reviews: Computational Statistics*, *3*(3), 282–287.

[12] Bergman B. (1986). Estimation of Weibull parameters using a weight function. *Journal of Materials Science Letters*, *5*(6), 611–614.

[13] Jia, X. (2021). A comparison of different least-squares methods for reliability of Weibull distribution based on right censored data. *Journal of Statistical Computation and Simulation*, *91*(5), 976–999.

[14] Jia, X. (2020). Reliability analysis for Weibull distribution with homogeneous heavily censored data based on Bayesian and least-squares methods. *Applied Mathematical Modelling*, *83*, 169–188.

[15] Lu, F., Huang, X., Lu, X., Tian, G., & Yang, J. (2023). Model detection for semiparametric accelerated failure additive model with right-censored data. *Statistical Methods in Medical Research*, *32*(8), 1527–1542.

[16] Zou, Y., Peng, Z., Cornell, J., Ye, P., & He, H. (2021). A new statistical test for latent class in censored data due to detection limit. *Statistics in Medicine*, *40*, 779–798.

[17] Tran, T. M. P., Abrams, S., Aerts, M., Maertens, K., & Hens, N. (2021). Measuring association among censored antibody titer data. *Statistics in Medicine*, *40*, 3740–3761.

[18] Jiang, W., Ye, Z., & Zhao, X. (2020). Reliability estimation from left-truncated and right-censored data using splines. *Statistica Sinica*, *30*(2), 845–875.

[19] Mcgough, S. F., Incerti, D., Lyalina, S., Copping, R., Narasimhan, B., & Tibshirani, R. (2021). Penalized regression for left-truncated and right-censored survival data. *Statistics in Medicine*, *40*(25), 5487–5500.

[20] Tang, N., Yan, X., & Zhao, X. (2020). Penalized generalized empirical likelihood with a diverging number of general estimating equations for censored data. *The Annals of Statistics*, *48*(1), 607–627.

[21] He, D., Zhou, Y., & Zou, H. (2020). High-dimensional variable selection with right censored length-biased data. *Statistica Sinica*, *30*, 193–215.

[22] Wu, D., & Li, C. (2021). Joint analysis of multivariate interval-censored survival data and a time-dependent covariate. *Statistical Methods in Medical Research*, *30*(3), 769–784.

[23] Valeriano, K. A., Galarza, C. E., & Matos, L. A. (2023). Moments and random number generation for the truncated elliptical family of distributions. *Statistics and Computing*, *33*(1), 32. https://doi.org/10.1007/s11222-022-10200-4

[24] Tan, Z. (2009). A new approach to MLE of Weibull distribution with interval data. *Reliability Engineering & System Safety*, *94*(2), 394–403.

[25] Nelson, W. B. (2003). *Applied Life Data Analysis*. Hoboken, NJ: John Wiley & Sons.

[26] Gibbons, D. I., & Vance, L. C. (1981). A simulation study of estimators for the 2-parameter Weibull distribution. *IEEE Transactions on Reliability*, *30*(1), 61–66.

[27] Genc, A., Erisoglu, M., Pekgor, A., Oturanc, G., Hepbasli, A., & Ulgen, K. (2005). Estimation of wind power potential using Weibull distribution. *Energy Sources*, *27*(9), 809–822.

[28] Pobocikova, I., Sedliackova, Z., Michalkova, M., & George, F. (2017). Monte Carlo comparison of the methods for estimating the Weibull distribution parameters–wind speed application. *Communications-Scientific Letters of the University of Zilina*, *19*(2A), 79–86.

[29] Joarder, A., Krishna, H., & Kundu, D. (2011). Inferences on Weibull parameters with conventional type-I censoring. *Computational Statistics & Data Analysis*, *55*(1), 1–11.

[30] Guo, J., Kong, X., Wu, N., & Xie, L. (2024). Evaluating the lifetime distribution parameters and reliability of products using successive approximation method. *Quality and Reliability Engineering International*, *40*, 3280–3303.

How to Cite: Yu, Y., Gong, J., & Zhu, K. (2025). Analytical Solution for Parameter Estimation of Weibull Distributions with Interval-Censored Data. *Journal of Data Science and Intelligent Systems*. https://doi.org/10.47852/bonviewJDSIS52024661