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Bonferroni Mean Operators Based on Interval-Valued Picture Hesitant Fuzzy Information and Their Application in Decision-Making Problems

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Abstract: In this article, we invent the Bonferroni mean (BM) operators using interval-valued picture hesitant fuzzy (IVPHF) technique, called IVPHF Bonferroni mean (IVPHFBM), IVPHF-weighted BM (IVPHFWBM), IVPHF geometric BM (IVPHFGBM), and IVPHF-weighted geometric BM (IVPHFWGBM) operators. These presented techniques are very beneficial and valuable because these are modified versions of many existing techniques. Moreover, we also examine three basic properties of each presented operator. In addition, we demonstrate the technique of multi-attribute decision-making (MADM) problem and try to describe it with the presence of evaluated techniques to show the capability and superiority of the invented theory. In last, we compare the prevailing techniques with presented studies to illustrate the supremacy and effectiveness of the derived approaches.

Keywords: picture fuzzy sets, hesitant fuzzy sets, interval-valued picture hesitant fuzzy sets, Bonferroni mean operators, multi-attribute decision-making techniques

1. Introduction

For depicting uncertain and unreliable information, the procedure of decision-making technique is famous. The multi-attribute decision-making (MADM) technique is the subpart of the decision-making procedure which is used for aggregating the finite family of information into a singleton set. For evaluating the finest optimal, many valuable and well-known applications have been proposed by different scholars such as artificial intelligence, machine learning, clustering analysis, and pattern recognition. In the consideration of the classical set theory, experts have lost a lot of data because of limited options such as zero or one. Therefore, to extend the range of options for a decision-maker, the theory of fuzzy set (FS) was established by Zadeh (1965). Moreover, the extended form of FS such as the fuzzy superior Mandelbrot set was derived by İnce and Ersoy (2022). Additionally, the interval-valued FS (IVFS) was examined by Zadeh (1975) by extending the range of the truth grade, because in the case of IVFS, experts have more possibility compared to simple FS. Moreover, in the presence of truth grade, we handle some problems but not at all; therefore, for evaluating maximum problems in real life, Atanassov (1986) exposed the idea of intuitionistic FS (IFS), where the IFS covered the truth and falsity grade. Furthermore, the interval-valued IFS (IVIFS) was evaluated by Atanassov (1999). The truth grade and falsity grade are not enough to evaluate many real-life problems. Therefore, the picture FS (PFS) was derived by Cuong (2013a) and Cuong (2013b) which covered the truth, abstinence, falsity, and neutral grades. Furthermore, Cuong (2014) modified the theory of PFS and derived the theory of interval-valued PFS (IVPFS). Additionally, Khalil et al. (2019) exposed the interval-valued picture fuzzy soft sets, Qin et al. (2016) examined the frank aggregation operators, and Lapo et al. (2022) evaluated the special types of UP-algebra.

FS has very famous and dominant and because of this reason many individuals have employed it in the environment of different fields, but some time experts have required the collection of truth grades instead of one grade because, during the voting system, candidates have needed a lot of votes in support instead of one vote in support. Therefore, Torra (2010) derived the hesitant FS (HFS) by extending the range of the truth grade, because in FS we have obtained one value again one element from a domain, but in the case of HFS, we obtained the collection of finite truth values against one element from a domain. Moreover, the interval-valued HFS (IVHFS) was invented by Farhadinia (2013), where the IVHFS is the modified version of the FSs, IVFSs, and HFSs. Moreover, Beg and Rashid (2014) invented the theory of intuitionistic HFS (IHFS), but the condition of IHFS has

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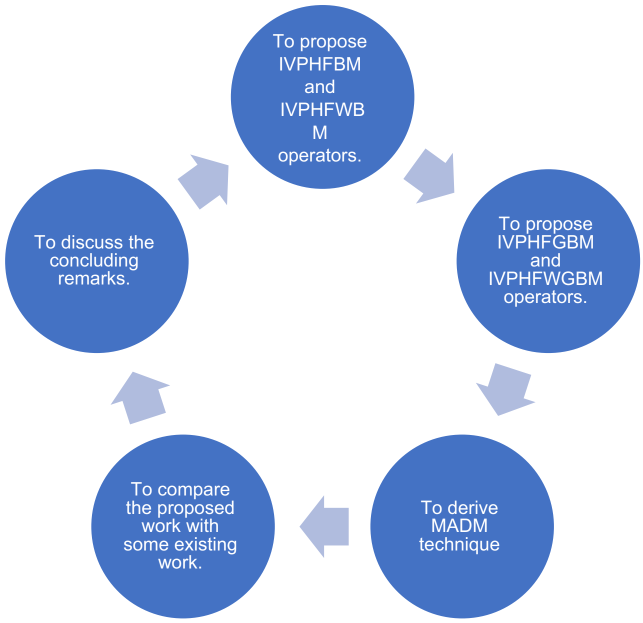
contained a lot of limitations and restrictions and because of this reason Mahmood et al. (2021b) updated the condition of IHFS and exposed the theory of improved IHFS and their application in decision-making problems. The Choquet integral-based TOPSIS technique for IHFS was derived by Joshi and Kumar (2016). Moreover, in two different papers, we have obtained the theory of picture HFS (PHFS) which was proposed by Wang and Li (2018) and Ullah et al. (2018). But the theory of interval-valued PHFS (IVPHFS) was derived by Khalil et al. (2019). Moreover, Kamaci et al. (2021) invented the theory of dynamic aggregation information in the presence of Einstein's operational laws for IVPHFS and their application. The IVPHFS is very beneficial, because they deal with many types of problems, for instance, FSs, HFSs, and interval-valued types of data. Furthermore, the existing types of data are the special cases of the IVPHFSs.

For aggregating the finite number of information into a singleton set, many scholars have derived different types of operators for FSs and their extensions, but the theory of Bonferroni mean (BM) operator and geometric BM (GBM) (Beliakov et al., 2010) is also very famous and reliable and aggregate the collection of data very easily. Moreover, the generalized BM operator for IFS was evaluated by Verma (2015), Xia et al. (2012), and Xu and Yager (2010). The extended BM operators for IFS were derived by Das et al. (2016). Additionally, the GBM operators for HFS were exposed by Zhu et al. (2012). Generalized BM operators for IVIFS were exposed by Rong et al. (2021) and Xu and Chen (2011). Ates and Akay (2020) derived the BM operators for PFSs and Mahmood et al. (2021a) examined the BM operators for PHFSs. Inspired by the above analysis, we noticed that the theory of the BM operator played a valuable role in aggregating the collection of information into a singleton set. Therefore, the major theme of this paper is listed below:

1. To expose the IVPHFBM operator and IVPHFWBM operator.
2. To derive the IVPHFGBM operator and IVPHFWGBM operator.
3. To examine three basic properties of each presented operator.
4. To demonstrate the technique of MADM problem and try to describe it with the presence of evaluated techniques to show the capability and superiority of the invented theory.
5. To compare the prevailing techniques with presented studies to illustrate the supremacy and effectiveness of the derived approaches. The graphical representation of the proposed work is stated in Figure 1.

This manuscript is summarized in the shape: In Section 2, we reviewed the major concept of IVPHFS, algebraic operational laws, BM operator, and GBM operator. In Section 3, we exposed the IVPHFBM operator, IVPHFWBM operator, IVPHFGBM operator, and IVPHFWGBM operator. In Section 4, we derived the MADM problem for proposed theory. In Section 5, we compared the prevailing techniques with presented studies to illustrate the supremacy and effectiveness of the derived approaches, where the concluding remarks are part of Section 6.

Figure 1
Geometrical abstract of the proposed work



2. Preliminaries

In this section, we reviewed the major concept of IVPHFS, algebraic operational laws, BM operator, and GBM operator in the consideration of a valuable universal set X.

Definition 1: [Mahmood et al., 2021a, 2021b] An IVPHFS $\mathfrak{F}$ based on universal set X is shown below:

$$\mathfrak{F} = \{\left(x, \left[M^{lp}_{\mathfrak{G}}(x), M^{up}_{\mathfrak{F}}(x)\right], \left[O^{lp}_{\mathfrak{F}}(x), O^{up}_{\mathfrak{F}}(x)\right], \left[N^{lp}_{\mathfrak{F}}(x), N^{up}_{\mathfrak{F}}(x)\right]\right) | x \in X\} \tag{1}$$

with $0 \leq \max\left(M^{up}_{\mathfrak{F}}(x)\right) + \max\left(O^{up}_{\mathfrak{F}}(x)\right) + \max\left(N^{up}_{\mathfrak{F}}(x)\right) \leq 1$, where based on the interval-valued truth grade $\left[M^{lp}_{\mathfrak{F}}(x), M^{up}_{\mathfrak{F}}(x)\right]$, interval-valued abstinence grade $\left[O^{lp}_{\mathfrak{F}}(x), O^{up}_{\mathfrak{F}}(x)\right]$, and interval-valued falsity grade $\left[N^{lp}_{\mathfrak{F}}(x), N^{up}_{\mathfrak{F}}(x)\right]$, we derived the refusal grade, such as: $R_{\mathfrak{F}}(x) = \left[R^{lp}_{\mathfrak{F}}(x), R^{up}_{\mathfrak{F}}(x)\right] = \left[1 - \max\left(M^{lp}_{\mathfrak{F}}(x)\right) + \max\left(O^{lp}_{\mathfrak{F}}(x)\right) + \max\left(N^{lp}_{\mathfrak{F}}(x)\right), 1 - \max\left(M^{up}_{\mathfrak{F}}(x)\right) + \max\left(O^{up}_{\mathfrak{F}}(x)\right) + \max\left(N^{up}_{\mathfrak{F}}\left(x\right)\right)\right]$. Finally, the simple form of IVPHFS is stated $\mathfrak{G}$y: $\mathfrak{F} = \left(\left[M^{lp}_{\mathfrak{F}}, M^{up}_{\mathfrak{F}}\right], \left[O^{lp}_{\mathfrak{F}}, O^{up}_{\mathfrak{F}}\right], \left[N^{lp}_{\mathfrak{F}}, N^{up}_{\mathfrak{F}}\right]\right)$.

Definition 2: [Mahmood et al., 2021a] Assumed any two IVPHFSs $\mathfrak{F} = \left(\left[M^{lp}_{\mathfrak{F}}, M^{up}_{\mathfrak{F}}\right], \left[O^{lp}_{\mathfrak{F}}, O^{up}_{\mathfrak{F}}\right], \left[N^{lp}_{\mathfrak{F}}, N^{up}_{\mathfrak{F}}\right]\right)$ and $\mathfrak{G} = \left(\left[M^{lp}_{\mathfrak{G}}, M^{up}_{\mathfrak{G}}\right], \left[O^{lp}_{\mathfrak{G}}, O^{up}_{\mathfrak{G}}\right], \left[N^{lp}_{\mathfrak{G}}, N^{up}_{\mathfrak{G}}\right]\right)$, then

$$\mathfrak{F} \cup \mathfrak{G} = \bigcup_{\begin{pmatrix} \alpha_1^{lp} \epsilon \mathrm{M}_{\mathrm{A}}^{lp} & \alpha_1^{up} \epsilon \mathrm{M}_{\mathrm{A}}^{up} \\ \alpha_2^{lp} \epsilon \mathrm{M}_{\mathfrak{G}}^{lp} & \alpha_2^{up} \epsilon \mathrm{M}_{\mathfrak{G}}^{up} \\ \phi_1^{lp} \epsilon \mathrm{O}_{\mathrm{A}}^{lp} & \phi_1^{up} \epsilon \mathrm{O}_{\mathrm{A}}^{up} \\ \phi_2^{lp} \epsilon \mathrm{O}_{\mathfrak{G}}^{lp} & , \phi_2^{up} \epsilon \mathrm{O}_{\mathfrak{G}}^{up} \\ \psi_1^{lp} \epsilon \mathrm{N}_{\mathrm{A}}^{lp} & \psi_1^{up} \epsilon \mathrm{N}_{\mathrm{A}}^{up} \\ \psi_2^{lp} \epsilon \mathrm{N}_{\mathfrak{G}}^{lp} & \psi_2^{up} \epsilon \mathrm{N}_{\mathfrak{G}}^{up} \end{pmatrix}} \left\{ \begin{array}{c} \left[\max\left(\varphi_1^{lp}, \varphi_2^{lp}\right), \max\left(\varphi_1^{up}, \varphi_2^{up}\right)\right], \left[\min\left(\phi_1^{lp}, \phi_2^{lp}\right), \min\left(\phi_1^{up}, \phi_2^{up}\right)\right], \\ \left[\min\left(\psi_1^{lp}, \psi_2^{lp}\right), \min\left(\psi_1^{up}, \psi_2^{up}\right)\right] \end{array} \right\} \tag{2}$$

$$\mathfrak{F} \cap \mathfrak{G} = \bigcup_{\begin{pmatrix} \varphi_1^{lp} \epsilon \mathrm{M}_{\mathrm{A}}^{lp} & \varphi_1^{up} \epsilon \mathrm{M}_{\mathrm{A}}^{up} \\ \varphi_2^{lp} \epsilon \mathrm{M}_{\mathfrak{G}}^{lp} & \varphi_2^{up} \epsilon \mathrm{M}_{\mathfrak{G}}^{up} \\ \phi_1^{lp} \epsilon \mathrm{O}_{\mathrm{A}}^{lp} & \phi_1^{up} \epsilon \mathrm{O}_{\mathrm{A}}^{up} \\ \phi_2^{lp} \epsilon \mathrm{O}_{\mathfrak{G}}^{lp} & , \phi_2^{up} \epsilon \mathrm{O}_{\mathfrak{G}}^{up} \\ \psi_1^{lp} \epsilon \mathrm{N}_{\mathrm{A}}^{lp} & \psi_1^{up} \epsilon \mathrm{N}_{\mathrm{A}}^{up} \\ \psi_2^{lp} \epsilon \mathrm{N}_{\mathfrak{G}}^{lp} & \psi_2^{up} \epsilon \mathrm{N}_{\mathfrak{G}}^{up} \end{pmatrix}} \left\{ \begin{array}{c} \left[\min\left(\varphi_1^{lp}, \varphi_2^{lp}\right), \min\left(\varphi_1^{up}, \varphi_2^{up}\right)\right], \left[\min\left(\phi_1^{lp}, \phi_2^{lp}\right), \min\left(\phi_1^{up}, \phi_2^{up}\right)\right], \\ \left[\max\left(\psi_1^{lp}, \psi_2^{lp}\right), \max\left(\psi_1^{up}, \psi_2^{up}\right)\right] \end{array} \right\} \tag{3}$$

$$\mathfrak{F} \oplus \mathfrak{G} = \bigcup_{\begin{pmatrix} \varphi_1^{lp} \epsilon \mathrm{M}_{\mathrm{A}}^{lp} & \varphi_1^{up} \epsilon \mathrm{M}_{\mathrm{A}}^{up} \\ \varphi_2^{lp} \epsilon \mathrm{M}_{\mathfrak{G}}^{lp} & \varphi_2^{up} \epsilon \mathrm{M}_{\mathfrak{G}}^{up} \\ \phi_1^{lp} \epsilon \mathrm{O}_{\mathrm{A}}^{lp} & \phi_1^{up} \epsilon \mathrm{O}_{\mathrm{A}}^{up} \\ \phi_2^{lp} \epsilon \mathrm{O}_{\mathfrak{G}}^{lp} & , \phi_2^{up} \epsilon \mathrm{O}_{\mathfrak{G}}^{up} \\ \psi_1^{lp} \epsilon \mathrm{N}_{\mathrm{A}}^{lp} & \psi_1^{up} \epsilon \mathrm{N}_{\mathrm{A}}^{up} \\ \psi_2^{lp} \epsilon \mathrm{N}_{\mathfrak{G}}^{lp} & \psi_2^{up} \epsilon \mathrm{N}_{\mathfrak{G}}^{up} \end{pmatrix}} \left\{ \begin{array}{c} \left[\varphi_1^{lp} + \varphi_2^{lp} - \varphi_1^{lp}\varphi_2^{lp}, \varphi_1^{up} + \varphi_2^{up} - \varphi_1^{up}\varphi_2^{up}\right], \\ \left[\phi_1^{lp}\phi_2^{lp}, \phi_1^{up}\phi_2^{up}\right], \left[\psi_1^{lp}\psi_2^{lp}, \psi_1^{up}\psi_2^{up}\right] \end{array} \right\} \tag{4}$$

$$\mathfrak{F} \otimes \mathfrak{G} = \bigcup_{\begin{pmatrix} \varphi_1^{lp} \epsilon \mathrm{M}_{\mathrm{A}}^{lp} & \varphi_1^{up} \epsilon \mathrm{M}_{\mathrm{A}}^{up} \\ \varphi_2^{lp} \epsilon \mathrm{M}_{\mathfrak{G}}^{lp} & \varphi_2^{up} \epsilon \mathrm{M}_{\mathfrak{G}}^{up} \\ \phi_1^{lp} \epsilon \mathrm{O}_{\mathrm{A}}^{lp} & \phi_1^{up} \epsilon \mathrm{O}_{\mathrm{A}}^{up} \\ \phi_2^{lp} \epsilon \mathrm{O}_{\mathfrak{G}}^{lp} & , \phi_2^{up} \epsilon \mathrm{O}_{\mathfrak{G}}^{up} \\ \psi_1^{lp} \epsilon \mathrm{N}_{\mathrm{A}}^{lp} & \psi_1^{up} \epsilon \mathrm{N}_{\mathrm{A}}^{up} \\ \psi_2^{lp} \epsilon \mathrm{N}_{\mathfrak{G}}^{lp} & \psi_2^{up} \epsilon \mathrm{N}_{\mathfrak{G}}^{up} \end{pmatrix}} \left\{ \begin{array}{c} \left[\varphi_1^{lp}\varphi_2^{lp}, \varphi_1^{up}\varphi_2^{up}\right], \left[\phi_1^{lp} + \phi_2^{lp} - \phi_1^{lp}\phi_2^{lp}, \phi_1^{up} + \phi_2^{up} - \phi_1^{up}\phi_2^{up}\right], \\ \left[\psi_1^{lp} + \psi_2^{lp} - \psi_1^{lp}\psi_2^{lp}, \psi_1^{up} + \psi_2^{up} - \psi_1^{up}\psi_2^{up}\right] \end{array} \right\} \tag{5}$$

$$\lambda\mathfrak{F} = \bigcup_{\begin{pmatrix} \varphi_1^{lp} \epsilon \mathrm{M}_{\mathrm{A}}^{lp} & \varphi_1^{up} \epsilon \mathrm{M}_{\mathrm{A}}^{up} \\ \varphi_2^{lp} \epsilon \mathrm{M}_{\mathfrak{G}}^{lp} & \varphi_2^{up} \epsilon \mathrm{M}_{\mathfrak{G}}^{up} \\ \phi_1^{lp} \epsilon \mathrm{O}_{\mathrm{A}}^{lp} & \phi_1^{up} \epsilon \mathrm{O}_{\mathrm{A}}^{up} \\ \phi_2^{lp} \epsilon \mathrm{O}_{\mathfrak{G}}^{lp} & , \phi_2^{up} \epsilon \mathrm{O}_{\mathfrak{G}}^{up} \\ \psi_1^{lp} \epsilon \mathrm{N}_{\mathrm{A}}^{lp} & \psi_1^{up} \epsilon \mathrm{N}_{\mathrm{A}}^{up} \\ \psi_2^{lp} \epsilon \mathrm{N}_{\mathfrak{G}}^{lp} & \psi_2^{up} \epsilon \mathrm{N}_{\mathfrak{G}}^{up} \end{pmatrix}} \left\{ \left[1 - \left(1 - \varphi_1^{lp}\right)^{\lambda}, 1 - \left(1 - \varphi_1^{up}\right)^{\lambda}\right], \left[\phi_1^{lp\lambda}, \phi_1^{up\lambda}\right], \left[\psi_1^{lp\lambda}, \psi_1^{up\lambda}\right] \right\} \tag{6}$$

$$\mathfrak{F}^{\lambda} = \bigcup_{\begin{pmatrix} \varphi_1^{lp} \epsilon \mathrm{M}_{\mathrm{A}}^{lp} & \varphi_1^{up} \epsilon \mathrm{M}_{\mathrm{A}}^{up} \\ \varphi_2^{lp} \epsilon \mathrm{M}_{\mathfrak{G}}^{lp} & \varphi_2^{up} \epsilon \mathrm{M}_{\mathfrak{G}}^{up} \\ \phi_1^{lp} \epsilon \mathrm{O}_{\mathrm{A}}^{lp} & \phi_1^{up} \epsilon \mathrm{O}_{\mathrm{A}}^{up} \\ \phi_2^{lp} \epsilon \mathrm{O}_{\mathfrak{G}}^{lp} & , \phi_2^{up} \epsilon \mathrm{O}_{\mathfrak{G}}^{up} \\ \psi_1^{lp} \epsilon \mathrm{N}_{\mathrm{A}}^{lp} & \psi_1^{up} \epsilon \mathrm{N}_{\mathrm{A}}^{up} \\ \psi_2^{lp} \epsilon \mathrm{N}_{\mathfrak{G}}^{lp} & \psi_2^{up} \epsilon \mathrm{N}_{\mathfrak{G}}^{up} \end{pmatrix}} \left\{ \begin{array}{c} \left[\varphi_1^{lp\lambda}, \varphi_1^{up\lambda}\right], \left[1 - \left(1 - \phi_1^{lp}\right)^{\lambda}, 1 - \left(1 - \phi_1^{up}\right)^{\lambda}\right], \\ \left[1 - \left(1 - \psi_1^{lp}\right)^{\lambda}, 1 - \left(1 - \psi_1^{up}\right)^{\lambda}\right] \end{array} \right\} \tag{7}$$

Definition 3: [Mahmood et al., 2021a] Assumed any IVPHFS $\mathfrak{F} = \left(\left[M^{lp}_{\mathfrak{F}}, M^{up}_{\mathfrak{F}}\right], \left[O^{lp}_{\mathfrak{F}}, O^{up}_{\mathfrak{F}}\right], \left[N^{lp}_{\mathfrak{F}}, N^{up}_{\mathfrak{F}}\right]\right)$, then

$$s(\mathfrak{F}) = \frac{1}{6}\left(\frac{1}{z}\sum_{\Xi=1}^{z}(M^{lp}_{\Xi}) + \frac{1}{z}\sum_{\Xi=1}^{z}(M^{up}_{\Xi}) - \frac{1}{m}\sum_{\Xi=1}^{m}(O^{lp}_{\Xi}) - \frac{1}{\mathfrak{d}}\sum_{\Xi=1}^{\mathfrak{d}}(N^{lp}_{\Xi}) - \frac{1}{m}\sum_{\Xi=1}^{m}(O^{up}_{\Xi}) - \frac{1}{\mathfrak{d}}\sum_{\Xi=1}^{\mathfrak{d}}(N^{up}_{\Xi})\right) \in [-1, 1] \tag{8}$$

$$h(\mathfrak{F}) = \frac{1}{6}\left(\frac{1}{z}\sum_{\Xi=1}^{z}(M^{lp}_{\Xi}) + \frac{1}{z}\sum_{\Xi=1}^{z}(M^{up}_{\Xi}) + \frac{1}{m}\sum_{\Xi=1}^{m}(O^{lp}_{\Xi}) + \frac{1}{\mathfrak{d}}\sum_{\Xi=1}^{\mathfrak{d}}(N^{lp}_{\Xi}) + \frac{1}{m}\sum_{\Xi=1}^{m}(O^{up}_{\Xi}) + \frac{1}{\mathfrak{d}}\sum_{\Xi=1}^{\mathfrak{d}}(N^{up}_{\Xi})\right) \in [0, 1] \tag{9}$$

noticed that $z, m, \mathfrak{d}$ stated the length of the $M_{\mathfrak{F}}, N_{\mathfrak{F}}, O_{\mathfrak{F}}$.

Definition 4: [Rong et al., 2021] Assume that the finite values of positive data $\sigma, \varsigma \geq 0$, $\mathfrak{F}_{\mathfrak{r}}(\mathfrak{r} = 1, 2, , \mathfrak{d})$, then we have the idea of the BM operator and GBM operator such as:

$$BM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{d}}) = \left(\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}\sum_{\substack{\Xi, j=1 \\ \Xi \neq j}}^{\mathfrak{d}}\left(\mathfrak{F}^{\sigma}_{\Xi}\mathfrak{F}^{\varsigma}_{j}\right)\right)^{\frac{1}{\sigma+\varsigma}} \tag{10}$$

$$GBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{d}}) = \frac{1}{\sigma+\varsigma}\left(\prod_{\substack{\Xi, j=1 \\ \Xi \neq j}}^{\mathfrak{d}}(\sigma\mathfrak{F}_{\Xi} + \varsigma\mathfrak{F}_j)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}} \tag{11}$$

3. BM/GBM Operators for IVPHFSs

In this section, we examined the theory of IVPHFBM, IVPHFWBM, IVPHFGBM, and IVPHFWGBM operators. Furthermore, we have stated their important properties such as idempotency, monotonicity, and boundedness.

Definition 5: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = \left(\left[M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, M^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right], \left[O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, O^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right], \left[N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, N^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right]\right)$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus, we stated the mathematical interpretation of IVPHFBM operator, such as:

$$IVPHFBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{d}}) = \left(\frac{1}{\mathfrak{d}(\mathfrak{d}-1)} \oplus^{\mathfrak{d}}_{\substack{\Xi, j=1 \\ \Xi \neq j}} \left(\mathfrak{F}^{\sigma}_{\Xi} \otimes \mathfrak{F}^{\varsigma}_{j}\right)\right)^{\frac{1}{\sigma+\varsigma}} \tag{12}$$

Theorem 1: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = \left(\left[M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, M^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right], \left[O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, O^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right], \left[N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, N^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right]\right)$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus, we derived that the aggregate data of equation (12) is an IVPHFS, such as:

$$IVPHFBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{d}}) = \bigcup_{\begin{pmatrix} \varphi^{lp}_{\Xi}\epsilon M^{lp}_{\Xi} & \varphi^{up}_{\Xi}\epsilon M^{up}_{\Xi} \\ \varphi^{lp}_{j}\epsilon M^{lp}_{j} & \varphi^{lp}_{j}\epsilon M^{lp}_{j} \\ \phi^{lp}_{\Xi}\epsilon O^{lp}_{\Xi} & \phi^{lp}_{\Xi}\epsilon O^{lp}_{\Xi} \\ \phi^{lp}_{j}\epsilon O^{lp}_{j} & \phi^{lp}_{j}\epsilon O^{lp}_{j} \\ \psi^{lp}_{\Xi}\epsilon N^{lp}_{\Xi} & \psi^{lp}_{\Xi}\epsilon N^{lp}_{\Xi} \\ \psi^{lp}_{j}\epsilon N^{lp}_{j} & \psi^{lp}_{j}\epsilon N^{lp}_{j} \end{pmatrix}} \left\{ \begin{array}{l} \left[\left(1 - \left(\prod_{\substack{\Xi, j=1 \\ \Xi \neq j}}^{\mathfrak{d}}\left(1 - \left(\varphi^{lp\sigma}_{\Xi}\varphi^{lp\varsigma}_{j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}, \left(1 - \left(\prod_{\substack{\Xi, j=1 \\ \Xi \neq j}}^{\mathfrak{d}}\left(1 - \left(\varphi^{up\sigma}_{\Xi}\varphi^{up\varsigma}_{j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right], \\ \left[\begin{array}{l} \left(1 - \left(1 - \left(\prod_{\substack{\Xi, j=1 \\ \Xi \neq j}}^{\mathfrak{d}}\left(\begin{array}{c} 2 - \left(1 - \phi^{lp}_{\Xi}\right)^{\sigma} - \left(1 - \phi^{lp}_{j}\right)^{\varsigma} - \\ \left(1 - \left(1 - \phi^{lp}_{\Xi}\right)^{\sigma}\right)\left(1 - \left(1 - \phi^{lp}_{j}\right)^{\varsigma}\right) \end{array}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right), \\ \left(1 - \left(1 - \left(\prod_{\substack{\Xi, j=1 \\ \Xi \neq j}}^{\mathfrak{d}}\left(\begin{array}{c} 2 - (1 - \phi^{up}_{\Xi})^{\sigma} - \left(1 - \phi^{up}_{j}\right)^{\varsigma} - \\ (1 - (1 - \phi^{up}_{\Xi})^{\sigma})\left(1 - \left(1 - \phi^{up}_{j}\right)^{\varsigma}\right) \end{array}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right) \end{array}\right], \\ \left[\begin{array}{l} \left(1 - \left(1 - \left(\prod_{\substack{\Xi, j=1 \\ \Xi \neq j}}^{\mathfrak{d}}\left(\begin{array}{c} 2 - \left(1 - \psi^{lp}_{\Xi}\right)^{\sigma} - \left(1 - \psi^{lp}_{j}\right)^{\varsigma} - \\ \left(1 - \left(1 - \psi^{lp}_{\Xi}\right)^{\sigma}\right)\left(1 - \left(1 - \psi^{lp}_{j}\right)^{\varsigma}\right) \end{array}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right), \\ \left(1 - \left(1 - \left(\prod_{\substack{\Xi, j=1 \\ \Xi \neq j}}^{\mathfrak{d}}\left(\begin{array}{c} 2 - (1 - \psi^{up}_{\Xi})^{\sigma} - \left(1 - \psi^{up}_{j}\right)^{\varsigma} - \\ (1 - (1 - \psi^{up}_{\Xi})^{\sigma})(1 - (1 - \psi^{up}_{j})^{\varsigma}) \end{array}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right) \end{array}\right] \end{array} \right\} \tag{13}$$

Moreover, we stated the important properties of the proposed data in equation (13).

Theorem 2: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = ([M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, M^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, O^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, N^{up}_{\mathfrak{F}_{\mathfrak{r}}}])$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus,

1. **Idempotency:** If $\mathfrak{F}_{\mathfrak{r}} = \mathfrak{F} = ([M^{lp}_{\mathfrak{F}}, M^{up}_{\mathfrak{F}}], [O^{lp}_{\mathfrak{F}}, O^{up}_{\mathfrak{F}}], [N^{lp}_{\mathfrak{F}}, N^{up}_{\mathfrak{F}}])$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$, then
$$IVPHFBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}}) = \mathfrak{F}. \tag{14}$$
2. **Monotonicity:** If $M'_{\mathfrak{r}} \leq M_{\mathfrak{r}}$,$O'_{\mathfrak{r}} \geq O_{\mathfrak{r}}$, $N'_{\mathfrak{r}} \geq N_{\mathfrak{r}}$, then
$$IVPHFBM^{\sigma,\varsigma}\left(\mathfrak{F}'_1, \mathfrak{F}'_2, , \mathfrak{F}'_{\mathfrak{d}}\right) \leq IVPHFBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}}). \tag{15}$$
3. **Boundedness:** If $\mathfrak{F}^+ = ([\max_{\mathfrak{r}} M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, \max_{\mathfrak{r}} M^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [\min_{\mathfrak{r}} O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, \min_{\mathfrak{r}} O^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [\min_{\mathfrak{r}} N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, \min_{\mathfrak{r}} N^{up}_{\mathfrak{F}_{\mathfrak{r}}}])$ and $\mathfrak{F}^- = ([\min_{\mathfrak{r}} M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, \min_{\mathfrak{r}} M^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [\min_{\mathfrak{r}} O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, \min_{\mathfrak{r}} O^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [\max_{\mathfrak{r}} N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, \max_{\mathfrak{r}} N^{up}_{\mathfrak{F}_{\mathfrak{r}}}])$, then
$$\mathfrak{F}^- \leq \mathrm{IVPHF}BM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}}) \leq \mathfrak{F}^+. \tag{16}$$

Definition 6: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = \left(\left[M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, M^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right], \left[O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, O^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right], \left[N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, N^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right]\right)$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus, we stated the mathematical interpretation of IVPHFWBM operator, such as:

$$\begin{aligned}&\mathrm{IVPHFW}BM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}})\\ &= \left(\frac{1}{\mathfrak{d}(\mathfrak{d}-1)} \sum_{\substack{\Xi, j = 1\\ \Xi \neq j}}^{\mathfrak{d}} (w_{\Xi}\mathfrak{F}_{\Xi})^{\sigma} \otimes \left(w_j \mathfrak{F}_j\right)^{\varsigma}\right)^{\frac{1}{\sigma+\varsigma}}\end{aligned} \tag{17}$$

where $w_j \in [0, 1], \sum_{j=1}^{\mathfrak{d}} w_j = 1$ represented the weight vector.

Theorem 3: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = \left(\left[M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, M^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right], \left[O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, O^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right], \left[N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, N^{up}_{\mathfrak{F}_{\mathfrak{r}}}\right]\right)$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus, we derived that the aggregate data of equation (17) is an IVPHFS, such as:

$$\mathrm{IVPHFW}BM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}}) = \bigcup_{\begin{pmatrix} \varphi^{lp}_{\Xi}\epsilon M^{lp}_{\Xi} & \varphi^{up}_{\Xi}\epsilon M^{up}_{\Xi}\\ \varphi^{lp}_{j}\epsilon M^{lp}_{j} & \varphi^{lp}_{j}\epsilon M^{lp}_{j}\\ \phi^{lp}_{\Xi}\epsilon O^{lp}_{\Xi} & \phi^{lp}_{\Xi}\epsilon O^{lp}_{\Xi}\\ \phi^{lp}_{j}\epsilon O^{lp}_{j} & \phi^{lp}_{j}\epsilon O^{lp}_{j}\\ \psi^{lp}_{\Xi}\epsilon N^{lp}_{\Xi} & \psi^{lp}_{\Xi}\epsilon N^{lp}_{\Xi}\\ \psi^{lp}_{j}\epsilon N^{lp}_{j} & \psi^{lp}_{j}\epsilon N^{lp}_{j}\end{pmatrix}} \left\{\begin{array}{l}
\left[\begin{array}{l}\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(1-\left(1-\varphi^{lp}_{\Xi}\right)^{w_{\Xi}}\right)^{\sigma}\left(1-\left(1-\varphi^{lp}_{j}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}},\\
\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-(1-(1-\varphi^{up}_{\Xi})^{w_{\Xi}})^{\sigma}\left(1-\left(1-\varphi^{up}_{j}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\end{array}\right],\\
\left[\begin{array}{l}\left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\begin{pmatrix}2-\left(1-\phi^{lp\,w_{\Xi}}_{\Xi}\right)^{\sigma}-\left(1-\phi^{lp\,w_j}_{j}\right)^{\varsigma}\\ -\left(1-\left(1-\phi^{lp\,w_{\Xi}}_{\Xi}\right)^{\sigma}\right)\left(1-\left(1-\phi^{lp\,w_{\Xi}}_{\Xi}\right)^{\varsigma}\right)\end{pmatrix}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right),\\
\left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\begin{pmatrix}2-(1-\phi^{up\,w_{\Xi}}_{\Xi})^{\sigma}-\left(1-\phi^{up\,w_j}_{j}\right)^{\varsigma}\\ -(1-(1-\phi^{up\,w_{\Xi}}_{\Xi})^{\sigma})(1-(1-\phi^{up\,w_{\Xi}}_{\Xi})^{\varsigma})\end{pmatrix}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right)\end{array}\right],\\
\left[\begin{array}{l}\left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\begin{pmatrix}2-\left(1-\psi^{lp\,w_{\Xi}}_{\Xi}\right)^{\sigma}-\left(1-\psi^{lp\,w_j}_{j}\right)^{\varsigma}\\ -\left(1-\left(1-\psi^{lp\,w_{\Xi}}_{\Xi}\right)^{\sigma}\right)\left(1-\left(1-\psi^{lp\,w_{\Xi}}_{\Xi}\right)^{\sigma}\right)\end{pmatrix}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right),\\
\left(1-\left(1-\left(\prod_{\substack{\Xi,\ j=1\\ \Xi\neq j}}^{\mathfrak{d}}\begin{pmatrix}2-(1-\psi^{up\,w_{\Xi}}_{\Xi})^{\sigma}-\left(1-\psi^{up\,w_j}_{j}\right)^{\varsigma}\\ -(1-(1-\psi^{up\,w_{\Xi}}_{\Xi})^{\sigma})(1-(1-\psi^{up\,w_{\Xi}}_{\Xi})^{\sigma})\end{pmatrix}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right)\end{array}\right]
\end{array}\right\}, \tag{18}$$

Definition 7: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = ([M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, M^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, O^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, N^{up}_{\mathfrak{F}_{\mathfrak{r}}}])$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus, we stated the mathematical interpretation of IVPHFGBM operator, such as:

$$\mathrm{IVPHFG}BM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}}) = \frac{1}{\sigma+\varsigma} \otimes^{\mathfrak{d}}_{\substack{\Xi, j = 1\\ \Xi \neq j}} \left(\sigma\mathfrak{F}_{\Xi} \oplus \varsigma\mathfrak{F}_{lp}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}} \tag{19}$$

Theorem 4: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = ([M^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, M^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [O^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, O^{up}_{\mathfrak{F}_{\mathfrak{r}}}], [N^{lp}_{\mathfrak{F}_{\mathfrak{r}}}, N^{up}_{\mathfrak{F}_{\mathfrak{r}}}])$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus, we derived that the aggregate data of equation (19) is an IVPHFS, such as:

$$
\begin{aligned}
&IVPHFGBM^{\sigma,\varsigma}(\mathfrak{F}_1,\mathfrak{F}_2,,\mathfrak{F}_{\mathfrak{d}}) \\
&= \bigcup_{\left(\begin{array}{ccc} \varphi_{\Xi}^{lp}\epsilon M_{\Xi}^{lp} & & \varphi_{\Xi}^{up}\epsilon M_{\Xi}^{up} \\ \varphi_{j}^{lp}\epsilon M_{j}^{lp} & & \varphi_{j}^{lp}\epsilon M_{j}^{lp} \\ \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} & & \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} \\ \phi_{j}^{lp}\epsilon O_{j}^{lp} & , & \phi_{j}^{lp}\epsilon O_{j}^{lp} \\ \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} & & \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} \\ \psi_{j}^{lp}\epsilon N_{j}^{lp} & & \psi_{j}^{lp}\epsilon N_{j}^{lp} \end{array}\right)}
\left\{\begin{array}{c}
\left[\begin{array}{c}
\left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(\begin{array}{c} 2-\left(1-\varphi_{\Xi}^{lp}\right)^{\sigma}-\left(1-\varphi_{j}^{lp}\right)^{\varsigma}- \\ \left(1-\left(1-\varphi_{\Xi}^{lp}\right)^{\sigma}\right)\left(1-\left(1-\varphi_{j}^{lp}\right)^{\varsigma}\right)\end{array}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right), \\
\left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(\begin{array}{c} 2-\left(1-\varphi_{\Xi}^{up}\right)^{\sigma}-\left(1-\varphi_{j}^{up}\right)^{\varsigma}- \\ \left(1-\left(1-\varphi_{\Xi}^{up}\right)^{\sigma}\right)\left(1-\left(1-\varphi_{j}^{up}\right)^{\varsigma}\right)\end{array}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right)
\end{array}\right], \\
\left[\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(\phi^{lp\sigma}_{\Xi}\phi^{lp\varsigma}_{j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}},\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(\phi^{up\sigma}_{\Xi}\phi^{up\varsigma}_{j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right], \\
\left[\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(\psi^{lp\sigma}_{\Xi}\psi^{lp\varsigma}_{j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}},\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(\psi^{up\sigma}_{\Xi}\psi^{up\varsigma}_{j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right]
\end{array}\right\}
\end{aligned}
\tag{20}
$$

Moreover, we stated the important properties of the proposed data in equation (20).

Theorem 5: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = \left(\left[M_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, M_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right], \left[O_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, O_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right], \left[N_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, N_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right]\right)$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus,

1. **Idempotency:** If $\mathfrak{F}_{\mathfrak{r}} = \mathfrak{F} = ([M_{\mathfrak{F}}^{lp}, M_{\mathfrak{F}}^{up}], [O_{\mathfrak{F}}^{lp}, O_{\mathfrak{F}}^{up}], [N_{\mathfrak{F}}^{lp}, N_{\mathfrak{F}}^{up}])$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$, then

$$IVPHFGBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}}) = \mathfrak{F}. \tag{21}$$

2. **Monotonicity:** If $M_{\mathfrak{r}}^{'} \leq M_{\mathfrak{r}}$,$O_{\mathfrak{r}}^{'} \geq O_{\mathfrak{r}}$, $N_{\mathfrak{r}}^{'} \geq N_{\mathfrak{r}}$, then

$$IVPHFGBM^{\sigma,\varsigma}\left(\mathfrak{F}_1^{'}, \mathfrak{F}_2^{'}, , \mathfrak{F}_{\mathfrak{d}}^{'}\right) \leq IVPHFGBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}}). \tag{22}$$

3. **Boundedness:** If $\mathfrak{F}^{+} = ([\max_{\mathfrak{r}} M_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, \max_{\mathfrak{r}} M_{\mathfrak{F}_{\mathfrak{r}}}^{up}], [\min_{\mathfrak{r}} O_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, \min_{\mathfrak{r}} O_{\mathfrak{F}_{\mathfrak{r}}}^{up}], [\min_{\mathfrak{r}} N_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, \min_{\mathfrak{r}} N_{\mathfrak{F}_{\mathfrak{r}}}^{up}])$ and $\mathfrak{F}^{-} = ([\min_{\mathfrak{r}} M_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, \min_{\mathfrak{r}} M_{\mathfrak{F}_{\mathfrak{r}}}^{up}], [\min_{\mathfrak{r}} O_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, \min_{\mathfrak{r}} O_{\mathfrak{F}_{\mathfrak{r}}}^{up}], [\max_{\mathfrak{r}} N_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, \max_{\mathfrak{r}} N_{\mathfrak{F}_{\mathfrak{r}}}^{up}])$, then

$$\mathfrak{F}^{-} \leq IVPHFGBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, , \mathfrak{F}_{\mathfrak{d}}) \leq \mathfrak{F}^{+}. \tag{23}$$

Definition 8: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = \left(\left[M_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, M_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right], \left[O_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, O_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right], \left[N_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, N_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right]\right)$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus, we stated the mathematical interpretation of IVPHFWGBM operator, such as:

$$
\begin{aligned}
&IVPHFWGBM^{\sigma,\varsigma}(\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_{\mathfrak{d}}) \\
&= \frac{1}{\sigma+\varsigma} \otimes^{\mathfrak{d}}_{\substack{\Xi,j=1\\ \Xi\neq j}} \left(\sigma(\mathfrak{F}_{\Xi})^{w_{\Xi}} \oplus \varsigma(\mathfrak{F}_{j})^{w_j}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}
\end{aligned}
\tag{24}
$$

Theorem 6: Assume any collection of IVPHFSs $\mathfrak{F}_{\mathfrak{r}} = \left(\left[M_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, M_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right], \left[O_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, O_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right], \left[N_{\mathfrak{F}_{\mathfrak{r}}}^{lp}, N_{\mathfrak{F}_{\mathfrak{r}}}^{up}\right]\right)$ $(\mathfrak{r} = 1, 2, , \mathfrak{d})$ with $\sigma, \varsigma \geq 0$. Thus, we derived that the aggregate data of equation (24) is an IVPHFS, such as:

$$
IVPHWGBM^{\sigma,\varsigma}(\mathfrak{F}_1,\mathfrak{F}_2,,\mathfrak{F}_{\mathfrak{d}}) = \bigcup_{\left(\begin{array}{ccc} \varphi_{\Xi}^{lp}\epsilon M_{\Xi}^{lp} & & \varphi_{\Xi}^{up}\epsilon M_{\Xi}^{up} \\ \varphi_{j}^{lp}\epsilon M_{j}^{lp} & & \varphi_{j}^{lp}\epsilon M_{j}^{lp} \\ \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} & & \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} \\ \phi_{j}^{lp}\epsilon O_{j}^{lp} & , & \phi_{j}^{lp}\epsilon O_{j}^{lp} \\ \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} & & \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} \\ \psi_{j}^{lp}\epsilon N_{j}^{lp} & & \psi_{j}^{lp}\epsilon N_{j}^{lp} \end{array}\right)}
\left\{\begin{array}{c}
\left[\begin{array}{c}
\left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(\begin{array}{c} 2-\left(1-\left(\varphi_{\Xi}^{lp}\right)^{w_{\Xi}}\right)^{\sigma}-\left(1-\left(\varphi_{j}^{lp}\right)^{w_j}\right)^{\varsigma}- \\ \left(1-\left(1-\left(\varphi_{\Xi}^{lp}\right)^{w_{\Xi}}\right)^{\sigma}\right)\left(1-\left(1-\left(\varphi_{j}^{lp}\right)^{w_j}\right)^{\varsigma}\right)\end{array}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right), \\
\left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(\begin{array}{c} 2-\left(1-\left(\varphi_{\Xi}^{up}\right)^{w_{\Xi}}\right)^{\sigma}-\left(1-\left(\varphi_{j}^{up}\right)^{w_j}\right)^{\varsigma}- \\ \left(1-\left(1-\left(\varphi_{\Xi}^{up}\right)^{w_{\Xi}}\right)^{\sigma}\right)\left(1-\left(1-\left(\varphi_{j}^{up}\right)^{w_j}\right)^{\varsigma}\right)\end{array}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}\right)
\end{array}\right], \\
\left[\begin{array}{c}
\left(1-\left(\prod_{\Xi,j=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\phi_{\Xi}^{lp}\right)^{w_{\Xi}}\right)^{\sigma}\left(1-\left(1-\phi_{j}^{lp}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}, \\
\left(1-\left(\prod_{\Xi,j=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\phi_{\Xi}^{up}\right)^{w_{\Xi}}\right)^{\sigma}\left(1-\left(1-\phi_{j}^{up}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}
\end{array}\right], \\
\left[\begin{array}{c}
\left(1-\left(\prod_{\Xi,j=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\psi_{\Xi}^{lp}\right)^{w_{\Xi}}\right)^{\sigma}\left(1-\left(1-\psi_{j}^{lp}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}, \\
\left(1-\left(\prod_{\Xi,j=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\psi_{\Xi}^{up}\right)^{w_{\Xi}}\right)^{\sigma}\left(1-\left(1-\psi_{j}^{up}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma+\varsigma}}
\end{array}\right]
\end{array}\right\}
\tag{25}
$$

Moreover, we stated the important properties of the proposed data in equation (25).
Moreover, in the presence of the parameters p and q, we have the following special cases, such as:

1. For $\varsigma \to 0$ in equation (25), then we have the following theory, such as:

$$IVPHFWGBM^{\sigma,0}(\mathfrak{F}_1,\mathfrak{F}_2,,\mathfrak{F}_{\mathfrak{d}}) = \bigcup_{\left(\begin{smallmatrix} \varphi_{\Xi}^{lp}\epsilon M_{\Xi}^{lp} & \varphi_{\Xi}^{up}\epsilon M_{\Xi}^{up} \\ \varphi_{j}^{lp}\epsilon M_{j}^{lp} & \varphi_{j}^{lp}\epsilon M_{j}^{lp} \\ \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} & \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} \\ \phi_{j}^{lp}\epsilon O_{j}^{lp}, & \phi_{j}^{lp}\epsilon O_{j}^{lp} \\ \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} & \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} \\ \psi_{j}^{lp}\epsilon N_{j}^{lp} & \psi_{j}^{lp}\epsilon N_{j}^{lp} \end{smallmatrix}\right)} \left\{ \begin{array}{c} \left[\begin{array}{c} \left(1-\left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\left(1-\left(\varphi_{\Xi}^{lp}\right)^{w_\Xi}\right)^{\sigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma}}\right), \\ \left(1-\left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\left(1-\left(\varphi_{\Xi}^{up}\right)^{w_\Xi}\right)^{\sigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma}}\right) \end{array} \right], \\ \left[\begin{array}{c} \left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\phi_{\Xi}^{lp}\right)^{w_\Xi}\right)^{\sigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma}}, \\ \left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\phi_{\Xi}^{up}\right)^{w_\Xi}\right)^{\sigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma}} \end{array} \right], \\ \left[\begin{array}{c} \left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\psi_{\Xi}^{lp}\right)^{w_\Xi}\right)^{\sigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma}}, \\ \left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\psi_{\Xi}^{up}\right)^{w_\Xi}\right)^{\sigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\sigma}} \end{array} \right] \end{array} \right\} \tag{26}$$

2. For $\sigma = 1, \varsigma \to 0$ in equation (25), then we have the following theory, such as:

$$IVPHFGBM^{1,0}(\mathfrak{F}_1,\mathfrak{F}_2,,\mathfrak{F}_{\mathfrak{d}}) = \bigcup_{\left(\begin{smallmatrix} \varphi_{\Xi}^{lp}\epsilon M_{\Xi}^{lp} & \varphi_{\Xi}^{up}\epsilon M_{\Xi}^{up} \\ \varphi_{j}^{lp}\epsilon M_{j}^{lp} & \varphi_{j}^{lp}\epsilon M_{j}^{lp} \\ \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} & \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} \\ \phi_{j}^{lp}\epsilon O_{j}^{lp}, & \phi_{j}^{lp}\epsilon O_{j}^{lp} \\ \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} & \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} \\ \psi_{j}^{lp}\epsilon N_{j}^{lp} & \psi_{j}^{lp}\epsilon N_{j}^{lp} \end{smallmatrix}\right)} \left\{ \begin{array}{c} \left[\left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(\varphi_{\Xi}^{lp}\right)^{w_\Xi}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right), \left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(\varphi_{\Xi}^{up}\right)^{w_\Xi}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)\right], \\ \left[\left(1-\left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\phi_{\Xi}^{lp}\right)^{w_\Xi}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)\right), \left(1-\left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\phi_{\Xi}^{up}\right)^{w_\Xi}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)\right)\right], \\ \left[\left(1-\left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\psi_{\Xi}^{lp}\right)^{w_\Xi}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)\right), \left(1-\left(1-\left(\prod_{\Xi=1}^{\mathfrak{d}}\left(1-\psi_{\Xi}^{up}\right)^{w_\Xi}\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)\right)\right] \end{array} \right\} \tag{27}$$

3. For $\sigma \to 0$ in equation (25), then we have the following theory, such as:

$$IVPHFWBM^{0,\varsigma}(\mathfrak{F}_1,\mathfrak{F}_2,,\mathfrak{F}_{\mathfrak{d}}) = \bigcup_{\left(\begin{smallmatrix} \varphi_{\Xi}^{lp}\epsilon M_{\Xi}^{lp} & \varphi_{\Xi}^{up}\epsilon M_{\Xi}^{up} \\ \varphi_{j}^{lp}\epsilon M_{j}^{lp} & \varphi_{j}^{lp}\epsilon M_{j}^{lp} \\ \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} & \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} \\ \phi_{j}^{lp}\epsilon O_{j}^{lp}, & \phi_{j}^{lp}\epsilon O_{j}^{lp} \\ \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} & \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} \\ \psi_{j}^{lp}\epsilon N_{j}^{lp} & \psi_{j}^{lp}\epsilon N_{j}^{lp} \end{smallmatrix}\right)} \left\{ \begin{array}{c} \left[\begin{array}{c} \left(1-\left(1-\left(\prod_{j=1}^{\mathfrak{d}}\left(1-\left(1-\left(\varphi_{j}^{lp}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\varsigma}}\right), \\ \left(1-\left(1-\left(\prod_{j=1}^{\mathfrak{d}}\left(1-\left(1-\left(\varphi_{j}^{up}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\varsigma}}\right) \end{array} \right], \\ \left[\begin{array}{c} \left(1-\left(\prod_{j=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\phi_{j}^{lp}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\varsigma}}, \\ \left(1-\left(\prod_{j=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\phi_{j}^{up}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\varsigma}} \end{array} \right], \\ \left[\begin{array}{c} \left(1-\left(\prod_{j=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\psi_{j}^{lp}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\varsigma}}, \\ \left(1-\left(\prod_{j=1}^{\mathfrak{d}}\left(1-\left(1-\left(1-\psi_{j}^{up}\right)^{w_j}\right)^{\varsigma}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{\varsigma}} \end{array} \right] \end{array} \right\} \tag{28}$$

4. For $\sigma = \varsigma = 1$ in equation (25), then we have the following theory, such as:

$$IVPHFWGM^{1,1}(\mathfrak{F}_1,\mathfrak{F}_2,,\mathfrak{F}_{\mathfrak{d}}) = \bigcup_{\left(\begin{smallmatrix} \varphi_{\Xi}^{lp}\epsilon M_{\Xi}^{lp} & \varphi_{\Xi}^{up}\epsilon M_{\Xi}^{up} \\ \varphi_{j}^{lp}\epsilon M_{j}^{lp} & \varphi_{j}^{lp}\epsilon M_{j}^{lp} \\ \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} & \phi_{\Xi}^{lp}\epsilon O_{\Xi}^{lp} \\ \phi_{j}^{lp}\epsilon O_{j}^{lp}, & \phi_{j}^{lp}\epsilon O_{j}^{lp} \\ \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} & \psi_{\Xi}^{lp}\epsilon N_{\Xi}^{lp} \\ \psi_{j}^{lp}\epsilon N_{j}^{lp} & \psi_{j}^{lp}\epsilon N_{j}^{lp} \end{smallmatrix}\right)} \left\{ \begin{array}{c} \left[\begin{array}{c} \left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(\varphi_{\Xi}^{lp\,w_\Xi}+\varphi_{j}^{lp\,w_j}-\varphi_{\Xi}^{lp\,w_\Xi}\varphi_{j}^{lp\,w_j}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{2}}, \\ \left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(\varphi_{\Xi}^{up\,w_\Xi}+\varphi_{j}^{up\,w_j}-\varphi_{\Xi}^{up\,w_\Xi}\varphi_{j}^{up\,w_j}\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{2}} \end{array} \right], \\ \left[\begin{array}{c} \left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(1-\left(1-\phi_{\Xi}^{lp}\right)^{w_\Xi}\right)\left(1-\left(1-\phi_{j}^{lp}\right)^{w_j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{2}}\right), \\ \left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(1-\left(1-\phi_{\Xi}^{up}\right)^{w_\Xi}\right)\left(1-\left(1-\phi_{j}^{up}\right)^{w_j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{2}}\right) \end{array} \right], \\ \left[\begin{array}{c} \left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(1-\left(1-\psi_{\Xi}^{lp}\right)^{w_\Xi}\right)\left(1-\left(1-\psi_{j}^{lp}\right)^{w_j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{2}}\right), \\ \left(1-\left(1-\left(\prod_{\substack{\Xi,j=1\\ \Xi\neq j}}^{\mathfrak{d}}\left(1-\left(1-\left(1-\psi_{\Xi}^{up}\right)^{w_\Xi}\right)\left(1-\left(1-\psi_{j}^{up}\right)^{w_j}\right)\right)\right)^{\frac{1}{\mathfrak{d}(\mathfrak{d}-1)}}\right)^{\frac{1}{2}}\right) \end{array} \right] \end{array} \right\} \tag{29}$$

To verify the proposed techniques, we illustrated some practical applications and tried to evaluate it with the help of invented theory.

4. MADM Technique for Proposed Methods

In this section, we demonstrated the MADM technique in the presence of the derived methods to show the validity and supremacy of the invented approaches.

For evaluating the above problem, we considered the collection of alternatives $\mathfrak{F}_1^{AL}, \mathfrak{F}_2^{AL}, \ldots, \mathfrak{F}_m^{AL}$ and for each alternative we have the collection of finite values of attributes $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ with weight vector $w_{\Xi} \in [0, 1]$ with a rule that is $\sum_{j=1}^{n} w_{\Xi} = 1$. Under the consideration of the above alternatives and attributes with weight vector, we compute an information matrix by including the collection of IVPHF values with $0 \leq \max(M_{\mathfrak{F}}^{up}(x)) + \max(O_{\mathfrak{F}}^{up}(x)) + max(N_{\mathfrak{F}}^{up}(x)) \leq 1$, where based on the interval-valued truth grade $[M_{\mathfrak{F}}^{lp}(x), M_{\mathfrak{F}}^{up}(x)]$, interval-valued abstinence grade $[O_{\mathfrak{F}}^{lp}(x), O_{\mathfrak{F}}^{up}(x)]$, and interval-valued falsity grade $[N_{\mathfrak{F}}^{lp}(x), N_{\mathfrak{F}}^{up}(x)]$, we derived the refusal grade, such as: $R_{\mathfrak{F}}(x) = [R_{\mathfrak{F}}^{lp}(x), R_{\mathfrak{F}}^{up}(x)] = [1 - \max(M_{\mathfrak{F}}^{lp}(x)) + \max(O_{\mathfrak{F}}^{lp}(x)) + max(N_{\mathfrak{F}}^{lp}(x)), 1 - \max(M_{\mathfrak{F}}^{up}(x)) + \max(O_{\mathfrak{F}}^{up}(x)) + max(N_{\mathfrak{F}}^{up}(x))]$. Finally, the simple form of IVPHFS is stated by: $\mathfrak{F} = ([M_{\mathfrak{F}}^{lp}, M_{\mathfrak{F}}^{up}], [O_{\mathfrak{F}}^{lp}, O_{\mathfrak{F}}^{up}], [N_{\mathfrak{F}}^{lp}, N_{\mathfrak{F}}^{up}])$. For depicting the above problem, we have the following procedure of decision-making such as:

Step 1: Arrange the data in the form of matrix by using the theory of IVPHF values.

Step 2: Normalize the information matrix, if needed, with the help of below theory, such as:

$$N = \begin{cases} \left(\left[M_{\mathfrak{F}}^{lp}, M_{\mathfrak{F}}^{up}\right], \left[O_{\mathfrak{F}}^{lp}, O_{\mathfrak{F}}^{up}\right], \left[N_{\mathfrak{F}}^{lp}, N_{\mathfrak{F}}^{up}\right]\right) & \textit{for benef}\,\Xi t \\ \left(\left[N_{\mathfrak{F}}^{lp}, N_{\mathfrak{F}}^{up}\right], \left[O_{\mathfrak{F}}^{lp}, O_{\mathfrak{F}}^{up}\right], \left[M_{\mathfrak{F}}^{lp}, M_{\mathfrak{F}}^{up}\right]\right) & \textit{for cost} \end{cases}$$

Step 3: Aggregate the information matrix with the help of IVPHFWBM operator and IVPHFWGBM operator.

Step 4: Evaluate the score values of the aggregated data.

Table 1
IVPHF decision matrix

	$\mathfrak{F}_1$	$\mathfrak{F}_2$
$\mathfrak{F}_1^{AL}$	$\left\{\begin{matrix}\{[0.1,0.2],[0.2,0.3]\},\\ \{[0.1,0.3],[0.2,0.3],[0.1,0.2],[0.2,0.4]\},\\ \{[0.1,0.2]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.11,0.21],[0.21,0.31]\},\\ \{[0.11,0.31],[0.21,0.31],[0.11,0.21],[0.21,0.41]\},\\ \{[0.11,0.21]\}\end{matrix}\right\}$
$\mathfrak{F}_2^{AL}$	$\left\{\begin{matrix}\{[0.2,0.3],[0.3,0.4]\},\\ \{[0.1,0.2],[0.1,0.2],[0.2,0.3],[0.1,0.3]\},\\ \{[0.1,0.2]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.21,0.31],[0.31,0.41]\},\\ \{[0.11,0.21],[0.11,0.21],[0.21,0.31],[0.11,0.31]\},\\ \{[0.11,0.21]\}\end{matrix}\right\}$
$\mathfrak{F}_3^{AL}$	$\left\{\begin{matrix}\{[0.2,0.2],[0.3,0.3]\},\\ \{[0.1,0.1],[0.2,0.2],[0.3,0.3],[0.1,0.2]\},\\ \{[0.2,0.3]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.21,0.21],[0.31,0.31]\},\\ \{[0.11,0.11],[0.21,0.21],[0.31,0.31],[0.11,0.21]\},\\ \{[0.21,0.31]\}\end{matrix}\right\}$
$\mathfrak{F}_4^{AL}$	$\left\{\begin{matrix}\{[0.2,0.4],[0.3,0.4]\},\\ \{[0.2,0.3],[0.1,0.3],[0.1,0.2],[0.2,0.2]\},\\ \{[0.1,0.1]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.21,0.41],[0.31,0.41]\},\\ \{[0.21,0.31],[0.11,0.31],[0.11,0.21],[0.21,0.21]\},\\ \{[0.11,0.11]\}\end{matrix}\right\}$
$\mathfrak{F}_5^{AL}$	$\left\{\begin{matrix}\{[0.4,0.5],[0.3,0.4]\},\\ \{[0.1,0.3],[0.2,0.3],[0.1,0.2],[0.2,0.3]\},\\ \{[0.1,0.1]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.41,0.51],[0.31,0.41]\},\\ \{[0.11,0.31],[0.21,0.31],[0.11,0.21],[0.21,0.31]\},\\ \{[0.11,0.11]\}\end{matrix}\right\}$
	$\mathfrak{F}_3$	$\mathfrak{F}_4$
$\mathfrak{F}_1^{AL}$	$\left\{\begin{matrix}\{[0.12,0.22],[0.22,0.32]\},\\ \left\{\begin{matrix}[0.12,0.32],[0.22,0.32],\\ [0.12,0.22],[0.22,0.42]\end{matrix}\right\},\\ \{[0.12,0.22]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.13,0.23],[0.23,0.33]\},\\ \{[0.13,0.33],[0.23,0.33],[0.13,0.23],[0.23,0.43]\},\\ \{[0.13,0.23]\}\end{matrix}\right\}$
$\mathfrak{F}_2^{AL}$	$\left\{\begin{matrix}\{[0.22,0.32],[0.32,0.42]\},\\ \left\{\begin{matrix}[0.12,0.22],[0.12,0.22],\\ [0.22,0.32],[0.12,0.32]\end{matrix}\right\},\\ \{[0.12,0.22]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.23,0.33],[0.33,0.43]\},\\ \{[0.13,0.23],[0.13,0.23],[0.23,0.33],[0.13,0.33]\},\\ \{[0.13,0.23]\}\end{matrix}\right\}$
$\mathfrak{F}_3^{AL}$	$\left\{\begin{matrix}\{[0.22,0.22],[0.32,0.32]\},\\ \left\{\begin{matrix}[0.12,0.12],[0.22,0.22],\\ [0.32,0.32],[0.12,0.22]\end{matrix}\right\},\\ \{[0.22,0.32]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.23,0.23],[0.33,0.33]\},\\ \{[0.13,0.13],[0.23,0.23],[0.33,0.33],[0.13,0.23]\},\\ \{[0.23,0.33]\}\end{matrix}\right\}$
$\mathfrak{F}_4^{AL}$	$\left\{\begin{matrix}\{[0.22,0.42],[0.32,0.42]\},\\ \left\{\begin{matrix}[0.22,0.32],[0.12,0.32],\\ [0.12,0.22],[0.22,0.22]\end{matrix}\right\},\\ \{[0.12,0.12]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.23,0.43],[0.33,0.43]\},\\ \{[0.23,0.33],[0.13,0.33],[0.13,0.23],[0.23,0.23]\},\\ \{[0.13,0.13]\}\end{matrix}\right\}$
$\mathfrak{F}_5^{AL}$	$\left\{\begin{matrix}\{[0.42,0.52],[0.32,0.42]\},\\ \left\{\begin{matrix}[0.12,0.32],[0.22,0.32],\\ [0.12,0.22],[0.22,0.32]\end{matrix}\right\},\\ \{[0.12,0.12]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.43,0.53],[0.33,0.43]\},\\ \{[0.13,0.33],[0.23,0.33],[0.13,0.23],[0.23,0.33]\},\\ \{[0.13,0.13]\}\end{matrix}\right\}$

Table 2
IVPHF aggregated matrix

	IVPHFWBM operator	IVPHFWGBM operator
$\mathfrak{F}_1^{AL}$	$\left\{\begin{matrix}\{[0.1724, 0.2364], [0.2364, 0.2878]\}, \\ \left\{\begin{matrix}[0.0302, 0.09], [0.0587, 0.09], [0.0302, 0.0587], \\ [0.0587, 0.125]\end{matrix}\right\}, \\ \{[0.0302, 0.0587]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.0302, 0.0587], [0.0587, 0.09]\}, \\ \left\{\begin{matrix}[0.1724, 0.2878], [0.2364, 0.2878], \\ [0.1724, 0.2364], [0.2364, 0.3325]\end{matrix}\right\}, \\ \{[0.1724, 0.2364]\}\end{matrix}\right\}$
$\mathfrak{F}_2^{AL}$	$\left\{\begin{matrix}\{[0.2364, 0.2878], [0.2878, 0.3325]\}, \\ \left\{\begin{matrix}[0.0302, 0.0587], [0.0302, 0.0587], \\ [0.0587, 0.09], [0.0302, 0.09]\end{matrix}\right\}, \\ \{[0.0302, 0.0587]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.0587, 0.09], [0.09, 0.125]\}, \\ \left\{\begin{matrix}[0.1724, 0.2364], [0.1724, 0.2364], \\ [0.2364, 0.2878], [0.1724, 0.2878]\end{matrix}\right\}, \\ \{[0.1724, 0.2364]\}\end{matrix}\right\}$
$\mathfrak{F}_3^{AL}$	$\left\{\begin{matrix}\{[0.2364, 0.2364], [0.2878, 0.2878]\}, \\ \left\{\begin{matrix}[0.0302, 0.0302], [0.0587, 0.0587], \\ [0.09, 0.09], [0.0302, 0.0587]\end{matrix}\right\}, \\ \{[0.0587, 0.09]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.0587, 0.0587], [0.09, 0.09]\}, \\ \left\{\begin{matrix}[0.1724, 0.1724], [0.2364, 0.2364], \\ [0.2878, 0.2878], [0.1724, 0.2364]\end{matrix}\right\}, \\ \{[0.2364, 0.2878]\}\end{matrix}\right\}$
$\mathfrak{F}_4^{AL}$	$\left\{\begin{matrix}\{[0.2364, 0.3325], [0.2878, 0.3325]\}, \\ \left\{\begin{matrix}[0.0587, 0.09], [0.0302, 0.09], \\ [0.0302, 0.0587], [0.0587, 0.0587]\end{matrix}\right\}, \\ \{[0.0302, 0.0302]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.0587, 0.125], [0.09, 0.125]\}, \\ \left\{\begin{matrix}[0.2364, 0.2878], [0.1724, 0.2878], \\ [0.1724, 0.2364], [0.2364, 0.2364]\end{matrix}\right\}, \\ \{[0.1724, 0.1724]\}\end{matrix}\right\}$
$\mathfrak{F}_5^{AL}$	$\left\{\begin{matrix}\{[0.3325, 0.3731], [0.2878, 0.3325]\}, \\ \left\{\begin{matrix}[0.0302, 0.09], [0.0587, 0.09], [0.0302, 0.0587], \\ [0.0587, 0.09]\end{matrix}\right\}, \\ \{[0.0302, 0.0302]\}\end{matrix}\right\}$	$\left\{\begin{matrix}\{[0.125, 0.1648], [0.09, 0.125]\}, \\ \left\{\begin{matrix}[0.1724, 0.2878], [0.2364, 0.2878], \\ [0.1724, 0.2364], [0.2364, 0.2878]\end{matrix}\right\}, \\ \{[0.1724, 0.1724]\}\end{matrix}\right\}$

Step 5: Rank all the alternatives based on score values to find the best one.

4.1. Illustrative example

A well-known private university want to appoint a vice chancellor for their university. For the post of vice chancellor, five candidates are called for interview $\mathfrak{F}_1^{AL}, \mathfrak{F}_2^{AL}, \mathfrak{F}_3^{AL}, \mathfrak{F}_4^{AL}, \mathfrak{F}_5^{AL}$. To select the best one, we have the following criteria such as:

$\mathfrak{F}_1$: PHD degree (minimum education).
$\mathfrak{F}_2$: More than 30 years of experience in teaching and more than 10 years of experience in administration.
$\mathfrak{F}_3$: More than a hundred publications.
$\mathfrak{F}_4$: Other things (personality, behavior, and performance)

Based on the above information, we concentrate to find the finest one form the collection of all. For depicting the above problem, we have the following procedure of decision-making such as:

Step 1: Arrange the data in the form of matrix by using the theory of IVPHF values, as shown in Table 1.

Step 2: Normalize the information matrix, if needed, with the help of below theory, such as:

$$N = \begin{cases} \left(\left[M_{\mathfrak{F}}^{lp}, M_{\mathfrak{F}}^{up}\right], \left[O_{\mathfrak{F}}^{lp}, O_{\mathfrak{F}}^{up}\right], \left[N_{\mathfrak{F}}^{lp}, N_{\mathfrak{F}}^{up}\right]\right) & for\ benef\Xi t \\ \left(\left[N_{\mathfrak{F}}^{lp}, N_{\mathfrak{F}}^{up}\right], \left[O_{\mathfrak{F}}^{lp}, O_{\mathfrak{F}}^{up}\right], \left[M_{\mathfrak{F}}^{lp}, M_{\mathfrak{F}}^{up}\right]\right) & for\ cost \end{cases}$$

But the data in Table 1 is not required to be normalized.

Step 3: Aggregate the information matrix with the help of IVPHFWBM operator and IVPHFWGBM operator, as shown in Table 2.

Step 4: Evaluate the score values of the aggregated data, as shown in Table 3.

Table 3
IVPHF score value matrix

	IVPHFWBM operator	IVPHFWGBM operator
$\mathfrak{F}_1^{AL}$	0.0404	−0.13
$\mathfrak{F}_2^{AL}$	0.064	−0.113
$\mathfrak{F}_3^{AL}$	0.044	−0.138
$\mathfrak{F}_4^{AL}$	0.0692	−0.102
$\mathfrak{F}_5^{AL}$	0.0793	−0.095

Table 4
IVPHF ranking matrix

Methods	IVPHFWBM operator
IVPHFWBM operator	$\mathfrak{F}_5^{AL} > \mathfrak{F}_4^{AL} > \mathfrak{F}_2^{AL} > \mathfrak{F}_3^{AL} > \mathfrak{F}_1^{AL}$
IVPHFWGBM operator	$\mathfrak{F}_5^{AL} > \mathfrak{F}_4^{AL} > \mathfrak{F}_2^{AL} > \mathfrak{F}_1^{AL} > \mathfrak{F}_3^{AL}$

Step 5: Rank all the alternatives based on score values to find the best one, as shown in Table 4.

The finest and best optimal is $\mathfrak{F}_5^{AL}$ in the presence of two different types of theory such as IVPHFWBM operator and IVPHFWGBM operator. Moreover, we stated the comparison between proposed techniques and some existing information to show the supremacy and validity of the derived theory.

5. Comparative Analysis

In this section, we concentrate on comparing the invented techniques with some prevailing techniques to show the validity and effectiveness of the derived theory. For this, we have used the following existing methods, such as power aggregation operators

Table 5
IVPHF comparative analysis

Methods	Score values	Ranking values
Mahmood et al. (2021a)	× × × × × × × ×	× × × × × × × ×
Joshi and Kumar (2016)	× × × × × × × ×	× × × × × × × ×
Ullah et al. (2018)	× × × × × × × ×	× × × × × × × ×
Khalil et al. (2019)	0.0593, 0.0792, 0.0657, 0.0848, 0.0955	$\mathfrak{F}_5^{AL} > \mathfrak{F}_4^{AL} > \mathfrak{F}_2^{AL} > \mathfrak{F}_3^{AL} > \mathfrak{F}_1^{AL}$
Kamaci et al. (2021)	−0.142, −0.129, −0.152, −0.12, −0.117	$\mathfrak{F}_5^{AL} > \mathfrak{F}_4^{AL} > \mathfrak{F}_2^{AL} > \mathfrak{F}_1^{AL} > \mathfrak{F}_3^{AL}$
IVPHFWBM operator	0.0404, 0.064, 0.044, 0.0692, 0.079	$\mathfrak{F}_5^{AL} > \mathfrak{F}_4^{AL} > \mathfrak{F}_2^{AL} > \mathfrak{F}_3^{AL} > \mathfrak{F}_1^{AL}$
IVPHFWGBM operator	−0.13, −0.113, −0.138, −0.102, −0.095	$\mathfrak{F}_5^{AL} > \mathfrak{F}_4^{AL} > \mathfrak{F}_2^{AL} > \mathfrak{F}_1^{AL} > \mathfrak{F}_3^{AL}$

for IHFS that was derived by Mahmood et al. (2021b), Choquet integral-based TOPSIS method for IVIHFS that was presented by Joshi and Kumar (2016), averaging aggregation operators for PHFS that was derived by Ullah et al. (2018), new aggregation operators for IVPHFS that was invented by Khalil et al. (2019), and the dynamic aggregation operators for IVPHFS that was exposed by Kamaci et al. (2021). Finally, using the data in Table 1, the comparison between presented operators and existing operators is stated in Table 5.

The finest and best optimal is $\mathfrak{F}_5^{AL}$ in the presence of two different types of theory such as IVPHFWBM operator and IVPHFWGBM operator (Khalil et al., 2019; Kamaci et al., 2021). Additionally, the proposed technique is new and valuable to cope with unreliable and vague data in real-life problems.

6. Conclusion

In this article, we evaluated the below information, such as:

1. We invented the IVPHFBM, IVPHFWBM, IVPHFGBM, and IVPHFBM operators.
2. These presented techniques are very beneficial and valuable because these are modified versions of many existing techniques.
3. We also examined three basic properties of each presented operator.
4. We demonstrated the technique of MADM problem and try to describe it with the presence of evaluated techniques to show the capability and superiority of the invented theory.
5. We compared the prevailing techniques with presented studies to illustrate the supremacy and effectiveness of the derived approaches.

In the upcoming times, we aim to employ the proposed work in the field of computer science, game theory, machine learning, artificial intelligence, clustering analysis, and decision-making (Saeed et al., 2023; Pérez-Canedo & Verdegay, 2023; Yazbek et al., 2023; Ullah, 2021; Akram et al., 2022) to enhance the worth of the invented theory.

Ethical Statement

This study does not contain any studies with human or animal subjects performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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