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Refined Fuzzy Soft Sets: Properties, Set-Theoretic Operations and Axiomatic Results

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Abstract: This article discusses the results of an investigation into refined fuzzy soft sets, a novel variant of traditional fuzzy sets. Refined fuzzy soft sets provide a versatile method of data analysis, inspired by the need to deal with uncertainty and ambiguity in real-world data. This research expands on prior work in fuzzy set theory by investigating the nature and characteristics of refined fuzzy soft sets. They are useful in decision-making, pattern recognition, image processing, and control theory because of their capacity to deal with uncertainty, ambiguity, and the inclusion of expert information. This study analyzes these fuzzy set models and compares them to others in the field to reveal their advantages and disadvantages. The practical uses of enhanced fuzzy soft sets are also examined, along with possible future research strategies on this exciting new topic.

Keywords: fuzzy set, soft set, fuzzy soft set, refined fuzzy soft set

1. Introduction

Fuzzy set theory is a mathematical paradigm for dealing with uncertainty and ambiguity in data and knowledge representation. Zadeh (1965) initially proposed the idea of a fuzzy set as a method of generalizing the standard definition of a set, which presupposes that a member either belongs to or does not belong to a set. A membership function that assigns a degree of membership between [0,1] represents a fuzzy set, on the other hand, which permits partial membership of an element in a set. Since its inception, fuzzy set theory has been used in a variety of domains, including control systems, decision-making, pattern recognition, image processing, and many more. Intuitionistic fuzzy sets, type-2 fuzzy sets, and fuzzy rough sets are all examples of more complicated models based on fuzzy set theory. Fuzzy set theory's adaptability and utility have led to its broad usage in a variety of real-world applications investigated by some authors (De et al., 2000; Feng et al., 2010; Garg & Rani, 2021; Karnik & Mendel, 2001).

Molodtsov (1999) was the first person to propose the idea of soft sets as a wholly novel mathematical instrument for resolving issues involving apprehensions about the future. According to Molodtsov's description from 1999, a soft set is a parametric family of subsets of the universal set, in which each member is regarded as a collection of approximation elements of the soft set. Voskoglou (2023) suggested a parametric decision-making approach employing soft sets and gray numerals, which extends the soft set method. Kharal (2010) noted the distance and similarity measures for soft sets. Xiao (2018) proposes a hybrid approach to using FSSs in decision-making that combines fuzzy preference relations analysis based on belief entropy with the Dempster-Shafer (D-S) evidential concept. Yang et al. (2013) presented the idea of multi-FSSs as well as the ways in which they can be used in decision-making. Chen et al. (2005) presented the parameterization reduction of soft sets as well as the applications. Utilizing Sanchez's technique, Jafar et al. (2020) investigated the use of soft set interactions and soft matrices in medical treatment. Numerous researchers are drawn to soft set theory due to its various implications for disciplines such as function smoothness, decision-making, statistical inference, data processing, measurement concept, predicting, and operations investigations (Dalkilic, 2021; Molodtsov, 2001; Peng, 2019; Xiao et al., 2009; Zou & Xiao, 2008).

The real world is fraught with inaccuracy, ambiguity, and uncertainty. In our everyday lives, we primarily interact with ambiguous notions rather than precise ones. Interacting with ambiguities is a major issue in many disciplines, including economics, medicine, social science, atmospheric science, and engineering. Numerous scholars are now engaged in feature vagueness in latest decades. Several classical speculations are well renowned and efficaciously model uncertainty, including fuzzy set concept (Zadeh, 1965), probability theory, vague set model (Gau & Buehrer, 1993), rough set theory (Pawlak et al., 1995), intuitionistic fuzzy set (Alaca et al., 2006), and interval-valued fuzzy set (Gorzałczany, 1987). The concept of fuzzy soft set (FSS) theory was initiated by Maji et al. (2001). Peng and Garg (2018) presented three methods to address the interval-valued fuzzy soft decision-making issue using weighted distance-based estimation,

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combinatorial distance-based evaluation, as well as similarity measures. The notion of parameterized FSSs, as well as decision-making, was postulated by Zhu and Zhan (2016). Zhao et al. (2017) displayed an innovative decision-making method based on intuitionistic FSSs. Das (2018) described weighted fuzzy soft multisets as well as decision-making. Some other scholars (Alcantud, 2022; Ejegwa & Agbetayo, 2023; Rahman et al., 2020; Saeed & Harl, 2023; Saeed et al., 2023; Voskoglou, 2023) also deliberated the fuzzy soft in distinct aspects.

Rahman et al. (2021) utilized a unique concept to classify the key components for refined intuitionistic fuzzy sets, such as subset, equal set, null set, and complement set, in addition to their basic set-theoretic functions, such as union, intersection, extended intersection, restricted union, restricted intersection, and restricted difference. Alkhazaleh (2017) suggested the idea of the n-valued refined neutrosophic soft set (n-VRNSs for short) as a categorization of neutrosophic soft sets and identified certain functions (notably subset, complement, union, intersection, AND, and OR operations) on n-valued refined neutrosophic soft set theory. However, Khalil et al. (2019) corrected the claims (3) and (4) of Alkhazaleh's Proposition 3.6 and initiated two innovative concepts describing "subset" and "equal" of n-VRNSs, in addition to certain evidence and associated propositions. Smarandache (2019) initiated the refined intuitionistic fuzzy set notion by further partitioning invitation and non-membership significance.

The concept of refined fuzzy soft sets ($\Re$FSSs) offers a promising framework for dealing with uncertainty and incomplete information in a wide range of applications. The development of $\Re$FSSs represents a significant advancement in the field of fuzzy set theory and offers exciting opportunities for solving complex problems in decision-making, pattern recognition, and other fields. By providing a more nuanced and comprehensive representation of uncertainty, $\Re$FSSs enable a deeper understanding of complex systems and phenomena. This, in turn, has the potential to lead to more effective and efficient solutions to real-world problems. With further research and development, the applications of $\Re$FSSs are likely to continue to expand, opening up new possibilities for innovation and discovery. Therefore, the pursuit of research in this area represents an important and meaningful opportunity for those seeking to make a significant contribution to the field of mathematics and its applications in various fields.

2. Preliminaries

2.1. Definition

According to the definition from Zadeh (1965), suppose $\breve{U}$ be a set of alternates. A fuzzy set over $\breve{U}$ is a set defined by a function $\daleth_A$ representing a mapping.

$$\daleth_A : \breve{U} \to [0, 1]$$

$\daleth_A$ is known as the membership function of A, and the value $\daleth_A(v)$ is called the grade of membership of $v \in \breve{U}$. The value represents the degree of v belonging to the fuzzy set A. Thus, a fuzzy set A over $\breve{U}$ can be represented as follows:

$$A = \{(v \backslash \daleth_A(v) : v \in \breve{U}, \daleth_A(v) \in [0, 1]\}.$$

Note that the set of all the fuzzy sets over $\breve{U}$ will be denoted by $F(\breve{U})$.

2.2. Example

Nasreen wants to purchase a washing machine for her clothing purpose. She has to evaluate a unique washing machine which has all the specifications of a standard machine. Suppose $\breve{U} = \{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4\}$ be different brands of machine such that

$\varkappa_1$ = WashPro.
$\varkappa_2$ = Clean Tech.
$\varkappa_3$ = Aqua Wash.
$\varkappa_4$ = Elite Wash.
$\varkappa_5$ = SmartSpin.
$E = \{\nu_1, \nu_2, \nu_3, \nu_4\}$ be a set of parameters such that
ν_1 = Expensive
ν_2 = Beautiful
ν_3 = Cheap
ν_4 = Very expensive

Then, the fuzzy set $\wp$ over $\breve{U}$ can be written as

$$\wp = \{< \varkappa_1, 0.6 >, < \varkappa_2, 0.4 >, < \varkappa_3, 0.9 >, < \varkappa_4, 0.32 >\}.$$

2.3. Definition

Based on the definition from Cagman et al. (2011), assume $\breve{U}$ is a universe set and a set of attributes E with respect to $\breve{U}$. Let $P(\breve{U})$ denote the power set of $\breve{U}$ and $A \subseteq E$. A pair (F, A) is called a soft set over $\breve{U}$, where F is a mapping given by F: $A \to P(\breve{U})$. In other words, a soft set (F, A) over $\breve{U}$ is a family of subsets of $\breve{U}$. For $v \in A$, F(v) may be considered as the set of v-approximate elements of the soft sets (F, A). Thus (F, A) is defined as

$$(F, A) = \{F(v) \in P(\breve{U}) : v \in E, F(v) = \phi \text{ if } v \notin A\}.$$

2.4. Example

Consider Example 2.2, the soft set F_A over $\breve{U}$ where $A = \{\nu_1, \nu_2, \nu_3\} \subset E$ is stated as

$$F_A = \{(\nu_1, \{\varkappa_1, \varkappa_2, \varkappa_3\}), (\nu_2, \{\varkappa_2, \varkappa_4\}), (\nu_3, \{\varkappa_1, \varkappa_4\})\}$$

2.5. Definition

Based on the definition from Cagman et al. (2011), an FSS $\mathfrak{H}_A$ over $\breve{U}$ is a set defined by a function $\daleth_A$

$$\daleth_A : E \to P(\breve{U}) \text{ such that } \daleth_A(v) = \phi \text{ if } v \notin A.$$

Here, $\daleth_A$ is called fuzzy approximate function of the FFS $\mathfrak{H}_A$, and the value $\daleth_A(v)$ is a set called ν-element of the FSS for all $v \in E$. Thus, an FSS $\mathfrak{H}_A$ over $\breve{U}$ can be shown by the set of ordered pairs.

$$\mathfrak{H}_A = \{(\nu, \daleth_A(v)) : v \in E, \daleth_A(v) \in P(\breve{U})\}.$$

Note that the set of all FSSs over $\breve{U}$ will be denoted by $PS(\breve{U})$.

2.6. Example

Consider Example 2.2, the FSS $\mathfrak{H}_A$ where $A = \{v_1, v_2, v_3\} \subset E$ over $\breve{U}$ stated as

$$\mathfrak{H}_A = \left\{ \left(\mathrm{v}_1, \left\{\frac{\varkappa_2}{0.9}, \frac{\varkappa_3}{0.4}\right\}\right), \left(\mathrm{v}_2, \frac{\varkappa_1}{0.2}, \frac{\varkappa_2}{0.5}\right), \left(\mathrm{v}_3, \left\{\frac{\varkappa_1}{0.2}, \frac{\varkappa_3}{0.4}, \frac{\varkappa_4}{0.1}\right\}\right) \right\}$$

3. Basic Notions of $\mathfrak{R}$FSS

In this segment, the elementary essential properties of $\Re$FSS are investigated with examples.

3.1. Definition

An $\Re$FSS $\beth_{A_i}$ over $\breve{U}$ is a set defined by a function $\daleth_{A_i}$ representing a mapping.

$$\daleth_{A_i} : \mathrm{E} \to \mathrm{P}(\breve{U}) \text{ such that } \daleth_{A_i}(\mathrm{v}) = \phi \text{ if } \mathrm{v} \notin \mathrm{A}.$$

Here, $\daleth_{A_i}$ is called fuzzy approximate function of the $\Re$FSS $\beth_{A_i}$. Thus, an $\Re$FSS $\beth_{A_i}$ over $\breve{U}$ can be represented by the set of ordered pairs.

$$\beth_{A_i} = \left\{ \left(v, \daleth_{A_i}(v)\right) : \forall\ \mathrm{v} \in \mathrm{E},\ \beth_{A_i}(\mathrm{v})\ \in \mathrm{P}(\breve{U}) \right\}.$$

A_i *where* $i = 2, 3, 4 \ldots .n$ and $n \in \mathrm{N}$. Throughout $i = 2$ in this paper for better understandings.

Note that the set of all $\Re$FSSs over $\breve{U}$ will be denoted by $\Re FS(\breve{U})$.

Here, $\beth_{A_1,A_2}, \beth_{B_1,B_2}$ and $\beth_{C_1,C_2}$ represents $\Re$FSSs with two memberships. One can extend the membership values to $3, 4, 5 \ldots n$.

3.2. Example

Consider Example 2.2, the $\Re$FSS $\beth_{A_1,A_2}$ where $A = \{v_1, v_2, v_3\} \subset E$ over $\breve{U}$ stated as

$$\beth_{A_1,A_2} = \left\{ \begin{array}{c} \left(\mathrm{v}_1, \left\{\frac{\varkappa_2}{<0.9, 0.1>}, \frac{\varkappa_4}{<0.5, 0.4>}\right\}\right), \\ \left(\mathrm{v}_2, \left\{\frac{\varkappa_4}{<0.5, 0.4>}, \frac{\varkappa_5}{<0.8, 0.5>}\right\}\right), \\ \left(\mathrm{v}_4, \left\{\frac{\varkappa_1}{<0.2, 0.8>}, \frac{\varkappa_3}{<0.4, 0.6>}, \frac{\varkappa_5}{<0.8, 0.5>}\right\}\right) \end{array} \right\}$$

3.3. Definition

$\Re$FSS Subset. Let $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ be two $\Re$FSS, then $\beth_{A_1,A_2} \breve{\subseteq} \beth_{B_1,B_2}$, if $\daleth_{A_1,A_2}(\mathrm{v}) \breve{\subseteq} \daleth_{B_1,B_2}(\mathrm{v})$ for all $\mathrm{v} \in \mathrm{E}$.

3.4. Remark

$\beth_{A_1,A_2} \breve{\subseteq} \beth_{B_1,B_2}$ does not apply the definition of the classical subset; that is every element of $\beth_{A_1,A_2}$ is an element of $\beth_{B_1,B_2}$.

3.5. Example

Considering data given in Example 2.2, let $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ be two $\Re$FSSs such that.

$$\beth_{A_1,A_2} = \left\{ \left(\mathrm{v}_2, \left\{\frac{\varkappa_2}{<0.2, 0.4>}\right\}\right), \left(\mathrm{v}_4, \left\{\frac{\varkappa_1}{<0.8, 0.3>}\right\}\right) \right\}$$

And

$$\beth_{B_1,B_2} = \left\{ \left(\mathrm{v}_2, \left\{\frac{\varkappa_2}{<0.6, 0.8>}, \frac{\varkappa_4}{<0.3, 0.6>}\right\}\right), \left(\mathrm{v}_4, \left\{\frac{\varkappa_1}{<0.6, 0.4>}, \frac{\varkappa_4}{<0.3, 0.6>}\right\}\right) \right\}$$

Then, for all $\mathrm{v} \in \mathrm{E} \daleth_{A_1,A_2}(\mathrm{v}) \breve{\subseteq} \daleth_{B_1,B_2}(\mathrm{v})$ is valid. Hence, $\beth_{A_1,A_2} \breve{\subseteq} \beth_{B_1,B_2}$

3.6. Definition

Empty $\Re$FSS. Let $\beth_{A_1,A_2} \in \Re FS(\breve{U})$. If $\beth_{A_1,A_2}(\mathrm{v}) = \phi$ for all $\mathrm{v} \in \mathrm{E}$, then $\beth_{A_1,A_2}$ is called an empty $\Re$FSS, denoted by $\beth_{\Phi 1,\Phi 2}$.

3.7. Definition

Let $\beth_{A_1,A_2} \in \Re\mathrm{FS}(\breve{U})$. If $\beth_{A_1,A_2}(\mathrm{v}) = \breve{U}$ for all $\mathrm{v} \in \mathrm{A}$, then $\beth_{A_1,A_2}$ is called A-universal $\Re$FSS, denoted by $\beth_{A_1,A_2}$.

If $\mathrm{A} = \mathrm{E}$, then the A-universal $\Re$FSS is called universal $\Re F$SS, denoted by $\beth_{\breve{\mathrm{E1,E2}}}$.

3.8. Example

Assume that $\breve{U} = \{\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4, \varkappa_5\}$ is a universal set and $\mathrm{E} = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$ is a set of all parameters.

If $\mathrm{A} = \{\mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}, \daleth_{A_1,A_2}(\mathrm{v}_2) = \left\{\frac{\varkappa_2}{<0.6, 0.4>}, \frac{\varkappa_4}{<0.3, 0.6>}\right\}, \daleth_{A_1,A_2}(\mathrm{v}_3) = \phi$ and $\daleth_{A_1,A_2}(\mathrm{v}_4) = \breve{U}$, then the $\Re F$SS, $\beth_{A_1,A_2}$ is written by

$$\beth_{A_1,A_2} = \left\{ \left(\mathrm{v}_2, \left\{\frac{\varkappa_2}{<0.6, 0.4>}, \frac{\varkappa_4}{<0.3, 0.6>}\right\}\right), (\mathrm{v}_4, \breve{U}) \right\}.$$

If $\mathrm{B} = \{\mathrm{v}_1, \mathrm{v}_3\}$, and $\daleth_{B_1,B_2}(\mathrm{v}_1) = \phi, \daleth_{B_1,B_2}(\mathrm{v}_3) = \phi$, then the $\Re F$SS $\beth_{B_1,B_2}$ is an $\Re F$SS empty, that is, $\beth_{A_1,A_2} = \beth_{\Phi}$.

If $\mathrm{C} = \{\mathrm{v}_1, \mathrm{v}_2\}, \daleth_{\mathrm{C}}(\mathrm{v}_1) = \breve{U}$, and $\daleth_{\mathrm{C}}(\mathrm{v}_2) = \breve{U}$, then the $\Re F$SS $\beth_{\mathrm{C}}$ is a C universal $\Re F$SS, that is, $\beth_{\mathrm{C}} = \beth_{\breve{\mathrm{C}}}$

3.9. Definition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2} \in \Re FS(\breve{U})$. Then, $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ are $\Re F$SS – equal, written as $\beth_{A_1,A_2} = \beth_{B_1,B_2}$, if and only if $\daleth_{A_1,A_2}(\mathrm{v}) = \daleth_{B_1,B_2}(\mathrm{v})$ for all $\mathrm{v} \in \mathrm{E}$.

3.10. Definition

Let $\beth_{A_1,A_2} \in \Re FS(\breve{U})$. Then, the complement $\beth^{\tilde{c}}_{A_1,A_2}$ of $\beth_{A_1,A_2}$ is an $\Re F$SS such that.

$$\daleth_{A_1,A_2^{\tilde{c}}}(\mathrm{v}) = \daleth^{\mathrm{c}}_{A_1,A_2}(\mathrm{v}),\ \text{for all } \mathrm{v} \in \mathrm{E},$$

where $\daleth^{\mathrm{c}}_{A_1,A_2}(\mathrm{v})$ is complement of the set $\daleth_{A_1,A_2}(\mathrm{v})$.

3.11. Example

Considering data given in Example 2.2, if we have $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ are two $\Re F$SS such as

$$\beth_{A_1,A_2} = \left\{ \left(v_2, \left\{ \frac{\varkappa_2}{<0.6,0.8>}, \frac{\varkappa_4}{<0.3,0.6>} \right\} \right), \left(v_4, \left\{ \frac{\varkappa_1}{<0.6,0.4>}, \frac{\varkappa_4}{<0.3,0.6>} \right\} \right) \right\}$$

And

$$\beth_{B_1,B_2} = \left\{ \left(v_2, \left\{ \frac{\varkappa_2}{<0.6,0.8>}, \frac{\varkappa_4}{<0.3,0.6>} \right\} \right), \left(v_4, \left\{ \frac{\varkappa_1}{<0.6,0.4>}, \frac{\varkappa_4}{<0.3,0.6>} \right\} \right) \right\}$$

As $\daleth_{A_1,A_2}(v) = \daleth_{B_1,B_2}(v)$ for all $v \in E$. Hence

$$\beth_{A_1,A_2} = \beth_{B_1,B_2}$$

Now, complement of $\beth_{A_1,A_2}$ is $\beth^C_{A_1,A_2}$.

$$\beth^C_{A_1,A_2} = \left\{ \left(v_2, \left\{ \frac{\varkappa_2}{<0.4,0.2>}, \frac{\varkappa_4}{<0.7,0.4>} \right\} \right), \left(v_4, \left\{ \frac{\varkappa_1}{<0.4,0.6>}, \frac{\varkappa_4}{<0.7,0.4>} \right\} \right) \right\}$$

Similarly, one can find complement of $\beth_{B_1,B_2}$.

4. Aggregation Operations of ℜFSS

This section describes the set-theoretic operations of ℜFSS by utilizing the data presented in Example 2.2.

4.1. Definition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2} \in \Re FS(\breve{U})$. Then, the union of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ denoted by $\beth_{A_1,A_2} \tilde{\cup} \beth_{B_1,B_2}$ is defined by its fuzzy approximate function.

$$\daleth_{A\tilde{\cup}B}(v) = \daleth_A(v) \cup \daleth_B(v) \text{ for all } v \in E$$

4.2. Example

Assuming data given in example 2.2, let

$$\beth_{A_1,A_2} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.26,0.15>}, \frac{\varkappa_2}{<0.12,0.2>} \right\} \right), \\ \left(v_2, \left\{ \frac{\varkappa_3}{<0.2,0.25>} \right\} \right), \\ \left(v_3, \left\{ \frac{\varkappa_4}{<0.1,0.8>}, \frac{\varkappa_5}{<0.3,0.4>} \right\} \right) \end{array} \right\}$$

and

$$\beth_{B_1,B_2} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_4}{<0.4,0.7>} \right\} \right), \\ \left(v_2, \left\{ \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_2}{<0.5,0.6>} \right\} \right), \\ \left(v_3, \left\{ \frac{\varkappa_2}{<0.5,0.6>}, \frac{\varkappa_5}{<0.3,0.6>} \right\} \right) \end{array} \right\}$$

Be two ℜFSSs. Then, the union of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ is given as

$$\beth_{A_1,A_2} \tilde{\cup}\ \beth_{B_1,B_2} = \left\{ \begin{array}{c} \left(v_1, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_2}{<0.12,0.2>}, \\ \frac{\varkappa_4}{<0.4,0.7>} \end{array} \right\} \right), \\ \left(v_2, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_2}{<0.5,0.6>}, \\ \frac{\varkappa_3}{<0.8,0.25>} \end{array} \right\} \right), \\ \left(v_3, \left\{ \begin{array}{c} \frac{\varkappa_2}{<0.5,0.6>}, \frac{\varkappa_4}{<0.1,0.8>}, \\ \frac{\varkappa_5}{<0.3,0.6>} \end{array} \right\} \right) \end{array} \right\}$$

Similarly, one can find union of more than two ℜFSSs.

4.3. Definition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2} \in \Re FS(\breve{U})$. Then, the restricted union of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ denoted by $\beth_{A_1,A_2} \tilde{\cup}_R \beth_{B_1,B_2}$ is given as

$$\beth_{A_1,A_2} \tilde{\cup}_R \beth_{B_1,B_2} = \beth_{A_1,A_2} \cap \beth_{B_1,B_2}$$

where

$$\daleth_{A_1,A_2}(v) = \max\left\{ \daleth_{A1}(v), \daleth_{A_2}(v) \right\} \text{ for all } v \in E$$

4.4. Example

Assuming data given in Example 2.2, let

$$\beth_{A_1,A_2} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.26,0.15>}, \frac{\varkappa_2}{<0.12,0.2>} \right\} \right), \\ \left(v_4, \left\{ \frac{\varkappa_3}{<0.2,0.25>} \right\} \right) \end{array} \right\}$$

and

$$\beth_{B_1,B_2} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_4}{<0.4,0.7>} \right\} \right), \\ \left(v_2, \left\{ \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_2}{<0.5,0.6>} \right\} \right), \\ \left(v_3, \left\{ \frac{\varkappa_2}{<0.5,0.6>}, \frac{\varkappa_5}{<0.3,0.6>} \right\} \right) \end{array} \right\}$$

Be two ℜFSSs. Then, the restricted union of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ is given as

$$\beth_{A_1,A_2} \tilde{\cup}_R \beth_{B_1,B_2} = \left\{ \left(v_1, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_2}{<0.12,0.2>}, \\ \frac{\varkappa_4}{<0.4,0.7>} \end{array} \right\} \right) \right\}.$$

4.5. Definition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2} \in \Re FS(\breve{U})$. Then, the intersection of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ denoted by $\beth_{A_1,A_2} \tilde{\cap} \beth_{B_1,B_2}$ is defined by its fuzzy approximate function.

$$\daleth_{A\cap B}(v) = \daleth_A(v) \cap \daleth_B(v) \text{ for all } v \in E.$$

4.6. Example

Considering data given in Example 2.2, let

$$\beth_{A_1,A_2} = \left\{ \begin{matrix} \left(\nu_1, \left\{\frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>}\right\}\right), \\ \left(\nu_2, \left\{\frac{\varkappa_3}{<0.2, 0.25>}\right\}\right), \\ \left(\nu_3, \left\{\frac{\varkappa_4}{<0.1, 0.8>}, \frac{\varkappa_5}{<0.3, 0.4>}\right\}\right) \end{matrix} \right\}$$

and

$$\beth_{B_1,B_2} = \left\{ \begin{matrix} \left(\nu_1, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_4}{<0.4, 0.7>}\right\}\right), \\ \left(\nu_2, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}\right\}\right), \\ \left(\nu_3, \left\{\frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_5}{<0.3, 0.6>}\right\}\right) \end{matrix} \right\}$$

Be two $\Re$FSSs. Then, the intersection of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ is given as

$$\beth_{A_1,A_2} \widetilde{\cap} \beth_{B_1,B_2} = \left\{ \begin{matrix} \left(\nu_1, \left\{ \begin{matrix} \frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>}, \\ \frac{\varkappa_4}{<0.4, 0.7>} \end{matrix} \right\}\right), \\ \left(\nu_2, \left\{ \begin{matrix} \frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}, \\ \frac{\varkappa_3}{<0.8, 0.25>} \end{matrix} \right\}\right), \\ \left(\nu_3, \left\{ \begin{matrix} \frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_4}{<0.1, 0.8>}, \\ \frac{\varkappa_5}{<0.3, 0.4>} \end{matrix} \right\}\right) \end{matrix} \right\}$$

Similarly, one can find intersection of more than two $\Re$FSSs.

4.7. Definition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2} \in \Re FS(\breve{U})$. Then, the restricted intersection of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ denoted by $\beth_{A_1,A_2} \widetilde{\cap}_\varepsilon \beth_{B_1,B_2}$ is as

$$\beth_{A_1,A_2} \widetilde{\cap}_\varepsilon \beth_{B_1,B_2} = \beth_{A_1,A_2} \cap \beth_{B_1,B_2}$$

where

$$\daleth_{A_1,A_2}(\nu) = \min\left\{\daleth_{A1}(v), \daleth_{A_2}(v)\right\} \text{ for all } v \in E.$$

4.8. Example

Assuming data given in Example 2.2, let

$$\beth_{A_1,A_2} = \left\{ \begin{matrix} \left(\nu_1, \left\{\frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>}\right\}\right), \\ \left(\nu_4, \left\{\frac{\varkappa_3}{<0.2, 0.25>}\right\}\right) \end{matrix} \right\}$$

and

$$\beth_{B_1,B_2} = \left\{ \begin{matrix} \left(\nu_1, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_4}{<0.4, 0.7>}\right\}\right), \\ \left(\nu_2, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}\right\}\right), \\ \left(\nu_3, \left\{\frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_5}{<0.3, 0.6>}\right\}\right) \end{matrix} \right\}$$

Be two $\Re$FSSs. Then, the restricted intersection of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ is given as

$$\beth_{A_1,A_2} \widetilde{\cap}_\varepsilon \beth_{B_1,B_2} = \left\{ \left(\nu_1, \left\{ \begin{matrix} \frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>}, \\ \frac{\varkappa_4}{<0.4, 0.7>} \end{matrix} \right\}\right) \right\}.$$

4.9. Definition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2} \in \Re FS(\breve{U})$. Then, the restricted difference of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ denoted by $\beth_{A_1,A_2} -_R \beth_{B_1,B_2}$ is as

$$\beth_{A_1,A_2} -_R \beth_{B_1,B_2} = \beth_{A_1,A_2} -_R \beth_{B_1,B_2}$$

4.10. Example

Suppose example given in 2.2, let

$$\beth_{A_1,A_2} = \left\{ \begin{matrix} \left(\nu_1, \left\{\frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>}\right\}\right), \\ \left(\nu_4, \left\{\frac{\varkappa_3}{<0.2, 0.25>}\right\}\right) \end{matrix} \right\}$$

and

$$\beth_{B_1,B_2} = \left\{ \begin{matrix} \left(\nu_1, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_4}{<0.4, 0.7>}\right\}\right), \\ \left(\nu_2, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}\right\}\right), \\ \left(\nu_3, \left\{\frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_5}{<0.3, 0.6>}\right\}\right) \end{matrix} \right\}$$

Be two $\Re$FSSs. Then, the restricted difference of $\beth_{A_1,A_2}$ and $\beth_{B_1,B_2}$ is given as

$$ℶ_{A_1,A_2} -_R ℶ_{B_1,B_2} = \left\{ \left(\nu_4, \left\{ \frac{\varkappa_3}{<0.2, 0.25>} \right\} \right) \right\}.$$

And

$$ℶ_{B_1,B_2} -_R ℶ_{A_1,A_2} = \left\{ \begin{array}{c} \left(\nu_2, \left\{ \frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>} \right\} \right), \\ \left(\nu_3, \left\{ \frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_5}{<0.3, 0.6>} \right\} \right) \end{array} \right\}.$$

It can be easily observed.

$$ℶ_{A_1,A_2} -_R ℶ_{B_1,B_2} \neq ℶ_{B_1,B_2} -_R ℶ_{A_1,A_2}$$

4.11. Proposition

Let $ℶ_{A_1,A_2}, ℶ_{B_1,B_2} \in \Re FS(\breve{U})$. Then

1. $ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{\breve{E1,E2}}$
2. $ℶ_{\phi} \widetilde{\subseteq} ℶ_{A_1,A_2}$
3. $ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{A_1,A_2}$
4. $ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{B_1,B_2}$ and $ℶ_{B_1,B_2} \widetilde{\subseteq} ℶ_{C_1,C_2} \Rightarrow ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{C_1,C_2}$

4.12. Example

For (1), assuming data from Example 2.2, let

$$ℶ_{A_1,A_2} = \left\{ \begin{array}{c} \left(\nu_1, \left\{ \frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>} \right\} \right), \\ \left(\nu_2, \left\{ \frac{\varkappa_3}{<0.2, 0.25>} \right\} \right) \end{array} \right\}$$

And $ℶ_{\breve{E1,E2}} = \breve{U}$

It can be observed that $ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{\breve{E1,E2}}$.

$$\text{for (2)}, ℶ_{\phi} = \left\{ \begin{array}{c} \left(\nu_1, \left\{ \frac{\varkappa_1}{<0,0>}, \frac{\varkappa_2}{<0,0>} \right\} \right), \left(\nu_2, \left\{ \frac{\varkappa_3}{<0,0>} \right\} \right), \\ \left(\nu_3, \left\{ \frac{\varkappa_1}{<0,0>} \right\} \right), \left(\nu_4, \left\{ \frac{\varkappa_5}{<0,0>} \right\} \right) \end{array} \right\}$$

It can be observed that $ℶ_{\phi} \widetilde{\subseteq} ℶ_{A_1,A_2}$ and $ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{A_1,A_2}$.

For (4),

$$ℶ_{B_1,B_2} = \left\{ \begin{array}{c} \left(\nu_1, \left\{ \frac{\varkappa_1}{<0.50, 0.30>}, \frac{\varkappa_2}{<0.24, 0.4>} \right\} \right), \\ \left(\nu_2, \left\{ \frac{\varkappa_3}{<0.4, 0.50>} \right\} \right) \end{array} \right\}$$

And

$$ℶ_{C_1,C_2} = \left\{ \begin{array}{c} \left(\nu_1, \left\{ \frac{\varkappa_1}{<0.75, 0.60>}, \frac{\varkappa_2}{<0.5, 0.6>} \right\} \right), \\ \left(\nu_2, \left\{ \frac{\varkappa_3}{<0.8, 0.7>} \right\} \right) \end{array} \right\}$$

It can be observed that

$$ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{B_1,B_2} \text{and} ℶ_{B_1,B_2} \widetilde{\subseteq} ℶ_{C_1,C_2} \Rightarrow ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{C_1,C_2}$$

4.13. Proposition

Let $ℶ_{A_1,A_2}, ℶ_{B_1,B_2}, ℶ_{C_1,C_2} \in \Re FS(\breve{U})$ Then,

1. $ℶ_{A_1,A_2} = ℶ_{B_1,B_2}$ and $ℶ_{B_1,B_2} = ℶ_{C_1,C_2} \Rightarrow ℶ_{A_1,A_2} = ℶ_{C_1,C_2}$

2. $ℶ_{A_1,A_2} \widetilde{\subseteq} ℶ_{B_1,B_2}$ and $ℶ_{B_1,B_2} \subseteq ℶ_{A_1,A_2} \Leftrightarrow ℶ_{A_1,A_2} = ℶ_{B_1,B_2}$

4.14. Proposition

Let $ℶ_{A_1,A_2} \in \Re FS(\breve{U})$. Then,

1. $\left(ℶ^{\tilde{c}}_{A_1,A_2} \right)^{\tilde{c}} = ℶ_{A_1,A_2}$

2. $ℶ^{\tilde{c}}_{\Phi} = ℶ_{\tilde{E}}$

4.15. Example

For (1), Let $ℶ_{A_1,A_2} \in \Re FS(\breve{U})$ such that

$$ℶ_{A_1,A_2} = \left\{ \begin{array}{c} \left(\nu_1, \left\{ \frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>} \right\} \right), \\ \left(\nu_2, \left\{ \frac{\varkappa_3}{<0.2, 0.25>} \right\} \right), \\ \left(\nu_3, \left\{ \frac{\varkappa_4}{<0.1, 0.8>}, \frac{\varkappa_5}{<0.3, 0.4>} \right\} \right) \end{array} \right\}$$

$$(ℶ_{A_1,A_2})^{\tilde{c}} = \left\{ \begin{array}{c} \left(\nu_1, \left\{ \frac{\varkappa_1}{<0.74, 0.85>}, \frac{\varkappa_2}{<0.88, 0.8>} \right\} \right), \\ \left(\nu_2, \left\{ \frac{\varkappa_3}{<0.8, 0.75>} \right\} \right), \\ \left(\nu_3, \left\{ \frac{\varkappa_4}{<0.9, 0.2>}, \frac{\varkappa_5}{<0.7, 0.6>} \right\} \right) \end{array} \right\}$$

$$((ℶ_{A_1,A_2})^{\tilde{c}})^{\tilde{c}} = \left\{ \begin{array}{c} \left(\nu_1, \left\{ \frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>} \right\} \right), \\ \left(\nu_2, \left\{ \frac{\varkappa_3}{<0.2, 0.25>} \right\} \right), \\ \left(\nu_3, \left\{ \frac{\varkappa_4}{<0.1, 0.8>}, \frac{\varkappa_5}{<0.3, 0.4>} \right\} \right) \end{array} \right\}$$

Hence proved.

$$\left(\beth^{\check{c}}_{A_1,A_2}\right)^{\check{c}} = \beth_{A_1,A_2}$$

4.16. Proposition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2}, \beth_{C_1,C_2} \in \Re FS(\breve{U})$. Then,

1. $\beth_{A_1,A_2} \widetilde{\cup}\, \beth_{A_1,A_2} = \beth_{A_1,A_2}$

2. $\beth_{A_1,A_2} \widetilde{\cup} \beth_{\Phi} = \beth_{A_1,A_2}$

3. $\beth_{A_1,A_2} \widetilde{\cup} \beth_{\tilde{E}} = \beth_{\tilde{E}}$

4. $\beth_{A_1,A_2} \widetilde{\cup}\, \beth_{B_1,B_2} = \beth_{B_1,B_2} \widetilde{\cup} \beth_{A_1,A_2}$

5. $\beth_{A_1,A_2} \widetilde{\cup}_R\, \beth_{B_1,B_2} = \beth_{B_1,B_2} \widetilde{\cup}_R \beth_{A_1,A_2}$

6. $\left(\beth_{A_1,A_2} \widetilde{\cup} \beth_{B_1,B_2}\right) \widetilde{\cup} \beth_{C_1,C_2} = \beth_{A_1,A_2} \widetilde{\cup} \left(\beth_{B_1,B_2} \widetilde{\cup} \beth_{C_1,C_2}\right)$

4.17. Example

For (5)

$$\left(\beth_{A_1,A_2} \widetilde{\cup} \beth_{B_1,B_2}\right) \widetilde{\cup} \beth_{C_1,C_2} = \beth_{A_1,A_2} \widetilde{\cup} \left(\beth_{B_1,B_2} \widetilde{\cup} \beth_{C_1,C_2}\right)$$

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2}, \beth_{C_1,C_2} \in \Re FS(\breve{U})$ such that

$$\beth_{A_1,A_2} = \left\{ \begin{array}{c} \left(v_1, \left\{\frac{\varkappa_1}{<0.26, 0.15>}, \frac{\varkappa_2}{<0.12, 0.2>}\right\}\right), \\ \left(v_2, \left\{\frac{\varkappa_3}{<0.2, 0.25>}\right\}\right), \\ \left(v_3, \left\{\frac{\varkappa_4}{<0.1, 0.8>}, \frac{\varkappa_5}{<0.3, 0.4>}\right\}\right) \end{array} \right\}$$

$$\beth_{B_1,B_2} = \left\{ \begin{array}{c} \left(v_1, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_4}{<0.4, 0.7>}\right\}\right), \\ \left(v_2, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}\right\}\right), \\ \left(v_3, \left\{\frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_5}{<0.3, 0.6>}\right\}\right) \end{array} \right\}$$

And

$$\beth_{C_1,C_2} = \left\{ \left(v_3, \left\{\frac{\varkappa_4}{<0.5, 0.6>}, \frac{\varkappa_5}{<0.2, 0.8>}\right\}\right) \right\}$$

$$\left(\beth_{A_1,A_2} \widetilde{\cup} \beth_{B_1,B_2}\right) = \left\{ \begin{array}{c} \left(v_1, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.12, 0.2>}, \\ \frac{\varkappa_4}{<0.4, 0.7>} \end{array} \right\}\right), \\ \left(v_2, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}, \\ \frac{\varkappa_3}{<0.8, 0.25>} \end{array} \right\}\right), \\ \left(v_3, \left\{ \begin{array}{c} \frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_4}{<0.1, 0.8>}, \\ \frac{\varkappa_5}{<0.3, 0.6>} \end{array} \right\}\right) \end{array} \right\}$$

$$\left(\beth_{A_1,A_2} \widetilde{\cup} \beth_{B_1,B_2}\right) \widetilde{\cup} \beth_{C_1,C_2} = \left\{ \begin{array}{c} \left(v_1, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.12, 0.2>}, \\ \frac{\varkappa_4}{<0.4, 0.7>} \end{array} \right\}\right), \\ \left(v_2, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}, \\ \frac{\varkappa_3}{<0.8, 0.25>} \end{array} \right\}\right), \\ \left(v_3, \left\{ \begin{array}{c} \frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_4}{<0.5, 0.8>}, \\ \frac{\varkappa_5}{<0.3, 0.8>} \end{array} \right\}\right) \end{array} \right\}$$

Now, R.H.S $\beth_{A_1,A_2} \widetilde{\cup} \left(\beth_{B_1,B_2} \widetilde{\cup} \beth_{C_1,C_2}\right)$

$$\beth_{B_1,B_2} \widetilde{\cup} \beth_{C_1,C_2} = \left\{ \begin{array}{c} \left(v_1, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_4}{<0.4, 0.7>}\right\}\right), \\ \left(v_2, \left\{\frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}\right\}\right), \\ \left(v_3, \left\{ \begin{array}{c} \frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_4}{<0.5, 0.6>}, \\ \frac{\varkappa_5}{<0.3, 0.8>} \end{array} \right\}\right) \end{array} \right\}$$

$$\beth_{A_1,A_2} \widetilde{\cup} \left(\beth_{B_1,B_2} \widetilde{\cup} \beth_{C_1,C_2}\right) = \left\{ \begin{array}{c} \left(v_1, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.12, 0.2>}, \\ \frac{\varkappa_4}{<0.4, 0.7>} \end{array} \right\}\right), \\ \left(v_2, \left\{ \begin{array}{c} \frac{\varkappa_1}{<0.3, 0.8>}, \frac{\varkappa_2}{<0.5, 0.6>}, \\ \frac{\varkappa_3}{<0.8, 0.25>} \end{array} \right\}\right), \\ \left(v_3, \left\{ \begin{array}{c} \frac{\varkappa_2}{<0.5, 0.6>}, \frac{\varkappa_4}{<0.5, 0.8>}, \\ \frac{\varkappa_5}{<0.3, 0.8>} \end{array} \right\}\right) \end{array} \right\}$$

Similarly, one can also prove the other properties of 4.16 proposition.

4.18. Proposition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2}, \beth_{C_1,C_2} \in \Re FS(\breve{U})$. Then,

1. $\beth_{A_1,A_2} \widetilde{\cap}\, \beth_{A_1,A_2} = \beth_{A_1,A_2}$

2. $\beth_{A_1,A_2} \widetilde{\cap}\, \beth_{\Phi} = \beth_{\Phi}$

3. $\beth_{A_1,A_2} \widetilde{\cap}\, \beth_{\tilde{E}} = \beth_{A_1,A_2}$

4. $\beth_{A_1,A_2} \widetilde{\cap}\, \beth_{B_1,B_2} = \beth_{B_1,B_2} \widetilde{\cap}\, \beth_{A_1,A_2}$

5. $\beth_{A_1,A_2} \widetilde{\cap}_{\varepsilon}\, \beth_{B_1,B_2} = \beth_{B_1,B_2} \widetilde{\cap}_{\varepsilon}\, \beth_{A_1,A_2}$

6. $\left(\beth_{A_1,A_2} \widetilde{\cap}\, \beth_{B_1,B_2}\right) \widetilde{\cap}\, \beth_{C_1,C_2} = \beth_{A_1,A_2} \widetilde{\cap} \left(\beth_{B_1,B_2} \widetilde{\cap}\, \beth_{C_1,C_2}\right)$

4.19. Remark

Let $\beth_{A_1,A_2} \in \Re FS(\breve{U})$. If $\beth_{A_1,A_2} \neq \beth_{\Phi}$ and $\beth_{A_1,A_2} \neq \beth_{\tilde{E}}$, then $\beth_{A_1,A_2} \widetilde{\cup} \beth^{\check{c}}_{A_1,A_2} \neq \beth_{\tilde{E}}$ and $\beth_{A_1,A_2} \widetilde{\cap} \beth^{\check{c}}_{A_1,A_2} \neq \beth_{\Phi}$.

4.20. Example

To prove (5),

$$\left(\beth_{A_1,A_2}\widetilde{\cap}\beth_{B_1,B_2}\right)\widetilde{\cap}\beth_{C_1,C_2} = \beth_{A_1,A_2}\widetilde{\cap}\left(\beth_{B_1,B_2}\widetilde{\cap}\beth_{C_1,C_2}\right)$$

Let $\beth_{A_1,A_2},\beth_{B_1,B_2},\beth_{C_1,C_2} \in \Re FS\left(\breve{U}\right)$ such that

$$\beth_{A_1,A_2} = \left\{\begin{array}{c} \left(\nu_1,\left\{\frac{\varkappa_1}{<0.26,0.15>},\frac{\varkappa_2}{<0.12,0.2>}\right\}\right), \\ \left(\nu_2,\left\{\frac{\varkappa_3}{<0.2,0.25>}\right\}\right), \\ \left(\nu_3,\left\{\frac{\varkappa_4}{<0.1,0.8>},\frac{\varkappa_5}{<0.3,0.4>}\right\}\right) \end{array}\right\}$$

$$\beth_{B_1,B_2} = \left\{\begin{array}{c} \left(\nu_1,\left\{\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_4}{<0.4,0.7>}\right\}\right), \\ \left(\nu_2,\left\{\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_2}{<0.5,0.6>}\right\}\right), \\ \left(\nu_3,\left\{\frac{\varkappa_2}{<0.5,0.6>},\frac{\varkappa_5}{<0.3,0.6>}\right\}\right) \end{array}\right\}$$

And

$$\beth_{C_1,C_2} = \left\{\left(\nu_3,\left\{\frac{\varkappa_4}{<0.5,0.6>},\frac{\varkappa_5}{<0.2,0.8>}\right\}\right)\right\}$$

Consider L.H.S $\left(\beth_{A_1,A_2}\widetilde{\cap}\beth_{B_1,B_2}\right)\widetilde{\cap}\beth_{C_1,C_2}$

$$\left(\beth_{A_1,A_2}\widetilde{\cap}\beth_{B_1,B_2}\right) = \left\{\begin{array}{c} \left(\nu_1,\left\{\begin{array}{c}\frac{\varkappa_1}{<0.26,0.15>},\frac{\varkappa_2}{<0.12,0.2>}, \\ \frac{\varkappa_4}{<0.4,0.7>}\end{array}\right\}\right), \\ \left(\nu_2,\left\{\begin{array}{c}\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_2}{<0.5,0.6>}, \\ \frac{\varkappa_3}{<0.2,0.25>}\end{array}\right\}\right), \\ \left(\nu_3,\left\{\begin{array}{c}\frac{\varkappa_2}{<0.5,0.6>},\frac{\varkappa_4}{<0.1,0.8>}, \\ \frac{\varkappa_5}{<0.3,0.4>}\end{array}\right\}\right) \end{array}\right\}$$

$$\left(\beth_{A_1,A_2}\widetilde{\cap}\beth_{B_1,B_2}\right)\widetilde{\cap}\beth_{C_1,C_2} = \left\{\begin{array}{c} \left(\nu_1,\left\{\begin{array}{c}\frac{\varkappa_1}{<0.26,0.15>},\frac{\varkappa_2}{<0.12,0.2>}, \\ \frac{\varkappa_4}{<0.4,0.7>}\end{array}\right\}\right), \\ \left(\nu_2,\left\{\begin{array}{c}\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_2}{<0.5,0.6>}, \\ \frac{\varkappa_3}{<0.2,0.25>}\end{array}\right\}\right), \\ \left(\nu_3,\left\{\begin{array}{c}\frac{\varkappa_2}{<0.5,0.6>},\frac{\varkappa_4}{<0.1,0.6>}, \\ \frac{\varkappa_5}{<0.2,0.4>}\end{array}\right\}\right) \end{array}\right\}$$

Now, R.H.S $\beth_{A_1,A_2}\widetilde{\cap}\left(\beth_{B_1,B_2}\widetilde{\cap}\beth_{C_1,C_2}\right)$

$$\beth_{B_1,B_2}\widetilde{\cap}\beth_{C_1,C_2} = \left\{\begin{array}{c} \left(\nu_1,\left\{\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_4}{<0.4,0.7>}\right\}\right), \\ \left(\nu_2,\left\{\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_2}{<0.5,0.6>}\right\}\right), \\ \left(\nu_3,\left\{\begin{array}{c}\frac{\varkappa_2}{<0.5,0.6>},\frac{\varkappa_4}{<0.5,0.6>}, \\ \frac{\varkappa_5}{<0.2,0.6>}\end{array}\right\}\right) \end{array}\right\}$$

$$\beth_{A_1,A_2}\widetilde{\cap}\left(\beth_{B_1,B_2}\widetilde{\cap}\beth_{C_1,C_2}\right) = \left\{\begin{array}{c} \left(\nu_1,\left\{\begin{array}{c}\frac{\varkappa_1}{<0.26,0.15>},\frac{\varkappa_2}{<0.12,0.2>}, \\ \frac{\varkappa_4}{<0.4,0.7>}\end{array}\right\}\right), \\ \left(\nu_2,\left\{\begin{array}{c}\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_2}{<0.5,0.6>}, \\ \frac{\varkappa_3}{<0.2,0.25>}\end{array}\right\}\right), \\ \left(\nu_3,\left\{\begin{array}{c}\frac{\varkappa_2}{<0.5,0.6>},\frac{\varkappa_4}{<0.1,0.6>}, \\ \frac{\varkappa_5}{<0.2,0.4>}\end{array}\right\}\right) \end{array}\right\}$$

Hence proved.

$$\left(\beth_{A_1,A_2}\widetilde{\cap}\beth_{B_1,B_2}\right)\widetilde{\cap}\beth_{C_1,C_2} = \beth_{A_1,A_2}\widetilde{\cap}\left(\beth_{B_1,B_2}\widetilde{\cap}\beth_{C_1,C_2}\right)$$

Similarly using example, one can also verify the above proposition the results.

4.21. Proposition

Let $\beth_{A_1,A_2},\beth_{B_1,B_2} \in \Re FS\left(\breve{U}\right)$. Then, De Morgan's laws are valid as follows:

1. $\left(\beth_{A_1,A_2}\widetilde{\cup}\beth_{B_1,B_2}\right)^{\tilde{c}} = \beth^{\tilde{c}}_{A_1,A_2}\widetilde{\cap}\beth^{\tilde{c}}_{B_1,B_2}$

2. $\left(\beth_{A_1,A_2}\widetilde{\cap}\beth_{B_1,B_2}\right)^{\tilde{c}} = \beth^{\tilde{c}}_{A_1,A_2}\widetilde{\cup}\beth^{\tilde{c}}_{B_1,B_2}$.

Proof. The proofs can be obtained by using the respective approximate functions. For all $v \in E$,

$$\begin{aligned}(1)\text{: } \daleth_{\left(A\widetilde{\cup}B\right)^{\tilde{c}}}(v) &= \daleth^{c}_{A\widetilde{\cup}B}(v) \\ &= \left(\daleth_A(v)\cup\daleth_B(v)\right)^c \\ &= \left(\daleth_A(v)\right)^c\cap\left(\daleth_B(v)\right)^c \\ &= \daleth^c_A(v)\cap\daleth^c_B(v) \\ &= \daleth_{A^{\tilde{c}}}(v)\cap\daleth_{B^{\tilde{c}}}(v) \\ &= \daleth_{A^{\tilde{c}}\widetilde{\cap}B^{\tilde{c}}}(v)\end{aligned}$$

The proof of (2) is similar.

4.22. Example

To prove $\left(\beth_{A_1,A_2}\widetilde{\cup}\beth_{B_1,B_2}\right)^{\tilde{c}} = \beth^{\tilde{c}}_{A_1,A_2}\widetilde{\cap}\beth^{\tilde{c}}_{B_1,B_2}$, we consider an example.

Let $\beth_{A_1,A_2},\beth_{B_1,B_2} \in \Re FS(\breve{U})$ such that

$$\beth_{A_1,A_2} = \left\{\begin{array}{c} \left(\nu_1,\left\{\frac{\varkappa_1}{<0.26,0.15>},\frac{\varkappa_2}{<0.12,0.2>}\right\}\right), \\ \left(\nu_2,\left\{\frac{\varkappa_3}{<0.2,0.25>}\right\}\right), \\ \left(\nu_3,\left\{\frac{\varkappa_4}{<0.1,0.8>},\frac{\varkappa_5}{<0.3,0.4>}\right\}\right) \end{array}\right\}$$

$$\beth_{B_1,B_2} = \left\{\begin{array}{c} \left(\nu_1,\left\{\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_4}{<0.4,0.7>}\right\}\right), \\ \left(\nu_2,\left\{\frac{\varkappa_1}{<0.3,0.8>},\frac{\varkappa_2}{<0.5,0.6>}\right\}\right), \\ \left(\nu_3,\left\{\frac{\varkappa_2}{<0.5,0.6>},\frac{\varkappa_5}{<0.3,0.6>}\right\}\right) \end{array}\right\}$$

Then, L.H.S

$$\beth_{A_1,A_2}\widetilde{\cup}\beth_{B_1,B_2} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_2}{<0.12,0.2>}, \frac{\varkappa_4}{<0.4,0.7>} \right\}\right), \\ \left(v_2, \left\{ \frac{\varkappa_1}{<0.3,0.8>}, \frac{\varkappa_2}{<0.5,0.6>}, \frac{\varkappa_3}{<0.8,0.25>} \right\}\right), \\ \left(v_3, \left\{ \frac{\varkappa_2}{<0.5,0.6>}, \frac{\varkappa_4}{<0.1,0.8>}, \frac{\varkappa_5}{<0.3,0.6>} \right\}\right) \end{array} \right\}$$

$$\left(\beth_{A_1,A_2}\widetilde{\cup}\beth_{B_1,B_2}\right)^{\tilde{c}} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.7,0.2>}, \frac{\varkappa_2}{<0.88,0.8>}, \frac{\varkappa_4}{<0.6,0.3>} \right\}\right), \\ \left(v_2, \left\{ \frac{\varkappa_1}{<0.7,0.2>}, \frac{\varkappa_2}{<0.5,0.4>}, \frac{\varkappa_3}{<0.2,0.75>} \right\}\right), \\ \left(v_3, \left\{ \frac{\varkappa_2}{<0.5,0.4>}, \frac{\varkappa_4}{<0.9,0.2>}, \frac{\varkappa_5}{<0.7,0.4>} \right\}\right) \end{array} \right\}$$

Now, R.H.S

$$\left(\beth_{A_1,A_2}\right)^{\tilde{c}} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.74,0.85>}, \frac{\varkappa_2}{<0.88,0.8>} \right\}\right), \\ \left(v_2, \left\{ \frac{\varkappa_3}{<0.8,0.75>} \right\}\right), \\ \left(v_3, \left\{ \frac{\varkappa_4}{<0.9,0.2>}, \frac{\varkappa_5}{<0.7,0.6>} \right\}\right) \end{array} \right\}$$

$$\left(\beth_{B_1,B_2}\right)^{\tilde{c}} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.7,0.2>}, \frac{\varkappa_4}{<0.6,0.3>} \right\}\right), \\ \left(v_2, \left\{ \frac{\varkappa_1}{<0.7,0.2>}, \frac{\varkappa_2}{<0.5,0.4>} \right\}\right), \\ \left(v_3, \left\{ \frac{\varkappa_2}{<0.5,0.4>}, \frac{\varkappa_5}{<0.7,0.4>} \right\}\right) \end{array} \right\}$$

$$\beth^{\tilde{c}}_{A_1,A_2}\widetilde{\cap}\beth^{\tilde{c}}_{B_1,B_2} = \left\{ \begin{array}{c} \left(v_1, \left\{ \frac{\varkappa_1}{<0.7,0.2>}, \frac{\varkappa_2}{<0.88,0.8>}, \frac{\varkappa_4}{<0.6,0.3>} \right\}\right), \\ \left(v_2, \left\{ \frac{\varkappa_1}{<0.7,0.2>}, \frac{\varkappa_2}{<0.5,0.4>}, \frac{\varkappa_3}{<0.2,0.75>} \right\}\right), \\ \left(v_3, \left\{ \frac{\varkappa_2}{<0.5,0.4>}, \frac{\varkappa_4}{<0.9,0.2>}, \frac{\varkappa_5}{<0.7,0.4>} \right\}\right) \end{array} \right\}$$

Hence proved, L.H.S.=R.H.S. Similarly, we can also prove second property.

4.23. Proposition

Let $\beth_{A_1,A_2}, \beth_{B_1,B_2}, \beth_{C_1,C_2} \in \Re FS\left(\breve{U}\right)$. Then,

1. $\beth_{A_1,A_2}\widetilde{\cup}\left(\beth_{B_1,B_2}\widetilde{\cap}\beth_{C_1,C_2}\right) = \left(\beth_{A_1,A_2}\widetilde{\cup}\beth_{B_1,B_2}\right)\widetilde{\cap}\left(\beth_{A_1,A_2}\widetilde{\cup}\beth_{C_1,C_2}\right)$

2. $\beth_{A_1,A_2}\widetilde{\cap}\left(\beth_{B_1,B_2}\widetilde{\cup}\beth_{C_1,C_2}\right) = \left(\beth_{A_1,A_2}\widetilde{\cap}\beth_{B_1,B_2}\right)\widetilde{\cup}\left(\beth_{A_1,A_2}\widetilde{\cap}_{C_1,C_2}\right)$

Proof. For all $v \in E$,

$$(1): \quad \daleth_{A\widetilde{\cup}\left(B\widetilde{\cap}C\right)}(v) = \daleth_A(v) \cup \daleth_{B\widetilde{\cap}C}(v)$$

$$= \daleth_A(v) \cup \left(\daleth_B(v) \cap \daleth_C(v)\right)$$

$$= \left(\daleth_A(v) \cup \daleth_B(v)\right) \cap \left(\daleth_A(v) \cup \daleth_C(v)\right)$$

$$= \daleth_{A\widetilde{\cup}B}(v) \cap \daleth_{A\widetilde{\cup}C}(v)$$

$$= \daleth_{\left(A\widetilde{\cup}B\right)\widetilde{\cap}\left(A\widetilde{\cup}C\right)}(v).$$

Likewise, the proof of (2) can be in a similar way.

5. Conclusion

In this paper, $\Re$FSSs represent a powerful and flexible tool for handling uncertainty and incomplete information in a wide range of applications. By allowing for both degrees of membership and non-membership, $\Re$FSSs provide a more nuanced and comprehensive representation of uncertainty than traditional fuzzy sets or soft sets alone. Moreover, the various operations that can be performed on $\Re$FSSs such as complement, union, intersection, and projection enable powerful tools for manipulation and analysis of these sets. With further research, it is likely that $\Re$FSSs will continue to find new and innovative applications in decision-making, image processing, pattern recognition, and many other fields.

Conflicts of Interest

Harish Garg is the editor-in-chief for *Journal of Computational and Cognitive Engineering*, and was not involved in the editorial review or the decision to publish this article. The authors declare that they have no conflicts of interest to this work.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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