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Maclaurin Symmetric Mean Aggregation Operators Based on Spherical Fuzzy Information and Application to Decision-Making

Lemnaouar Zedam[1], Kifayat Ullah[2,*], Hafiz Muhammad Sajjad[2], Amir Hussain[2] and Azra Parveen[2,3]

[1]*Department of Data Analysis and Mathematical Modeling, University of Ghent, Belgium*

[2]*Department of Mathematics, Riphah International University, Pakistan*

[3]*Department of Business Administration, Bahauddin Zakariya University, Pakistan*

Abstract: In contemporary information fusion theory, the Maclaurin symmetric mean (MSM) operator is a traditional mean type aggregation operator (AO) that is an appropriate-able technique for aggregating numerical quantities. The MSM operator's ability to record the relationships between the several input arguments is one of its standout features. The spherical fuzzy set (SFS) is also a remarkable technique that covers the maximum information from real-life scenarios with the help of four grades. This manuscript consists of the development of the MSM and weighted MSM for the information obtained by SFS. Consequently, the spherical fuzzy MSM (SFMSM) and spherical fuzzy weighted MSM (SFWMSM) operators are developed, and their basic properties are studied. Finally, the developed SFMSM and SFWMSM operators have been applied to the real-life problem of the multi-attribute decision-making problem. All the results are compared and then clearly tabulated and graphed.

Keywords: Maclaurin symmetric mean, aggregation of information, spherical fuzzy information

1. Introduction

SFS is the framework that is used to extract information from any real-life scene. Moreover, the MSM operator is the AO that is used to aggregate the information by keeping the relation between attributes conservative. Hence, the MSM has been extended to the SFSs and a new technique has been developed.

The most familiar type of decision-making (DM), known as multi-attribute decision-making (MADM), seeks to choose the best option from a group of options when there are several, frequently at odds criteria. Due to its simplicity of use and acceptance as an important technique, MADM is used when DM themes and situations are indefinite. Zadeh (1965) proposed the idea of fuzzy sets (FSs), which describe the membership degree (MD) of data lies in [0,1] and provides an elastic stand to grip uncertainties, to solve such situations when the information contains uncertainty. By pairing MD with a non-membership degree (NMD) with the limitation that the total of both lies in [0,1], Atanassov (1986, 2000) introduced the intuitionistic FS (IFS). Even though IFS offers a better description of an object to real-life scenarios, it still limits the choice-maker's options and offers very little scope. To improve the IFSs limits and offer additional adaptable grounds for taking MDs and NMDs, Yager (2013, 2016) introduced the models of Pythagorean FS (PyFS) and q-rung orthopair FS (q-ROPFS). By enhancing the model of interval-valued FS (IVFS), which was investigated by Zadeh (1975) and Atanassov (1999) developed the model of interval-valued IFS (IVIFS) while keeping in mind the benefits of defining the MD and NMD in the shape of intervals rather than crisp values from [0,1]. Joshi et al. (2018) introduced valued q-ROPFS (IVq-ROPFS) in their paper.

The concept of IFS, PyFS, and q-ROPFS deals with complex and uncertain material in several real-world MADM and design gratitude circumstances, but these frameworks only discoursed the MD and NMD phases of human opinion, ignoring the abstinence degree (AD) and refusal degree (RD), which results in information loss. The concept of picture FS (PFS) constructed on MD, AD, NMD, and RD of data with the restriction that the total of all values must be in $[0, 1]$ interval was proposed by Cuong and Kreinovich (2014) to address such challenges. Furthermore, Cuong and Kreinovich (2014) developed the concept of interval-valued PFS (IVPFS). The tight constraint of PFS was loosened by establishing the idea of spherical FS (SFS) (Mahmood et al., 2019).

As, the AOs have been utilizing as the helpful tools for information collection and its aggregation. They are widely used

*__Corresponding author:__ Kifayat Ullah, Department of Mathematics, Riphah International University, Pakistan. Email: kifayat.khan.dr@gmail.com

in various branches of science. For example, in Biswas and Deb (2021) the family of some AOs is introduced and used to solve problems involving DM. Due to their applicability in several fields such as DM, the measurement of the continuous domain, and medical diagnosis, these AOs are very important.

The Maclaurin symmetric mean (MSM) operators are an essential component of aggregation techniques. Initially proposed by Maclaurin (1729), the MSM was later advanced by Gou et al. (2016). The interaction between the multi-input arguments is one of the MSM's standout characteristics. To handle MADM where the characteristics are enlightened, the MSM operator can offer information fusion that is more flexible and resilient. Additionally, the MSM operator decreases monotonically concerning restriction values of a provided set of arguments, which may represent the decision-maker's preferences for risk in actual solutions. The MSM has drawn increasing interest in recent years, and numerous significant results in theory and practice have been produced and can be found in Ren et al. (2016) and Zeng et al. (2016). Some more AOs can be found in Ali and Naeem (2022) and Chen (2014).

As we can see from the foregoing, the majority of the prevailing SF combination operators rely on the numerical sum of SFSs to perform the aggregation method and do not take into account the relationships between the multi-input elements. Many researchers have recently focused on the MSM operator and its implementations, which just take into account the scenario when arguments take the form of distinct values. In Garg (2016) and Maclaurin (1729), MSM AOs recount more than two input opinions, unlike conventional AOs. The concept of IF MSM operators was proposed by Reformat and Yager (2014) and further generalized by Liu (2017). Wang and Liu (2019) studied the use of PyFMSM operatives in DM. Wei et al. (2019) introduced the q-ROPFS MSM concept and used it in MADM. Moreover, Ullah (2021) generalized MSM for framework of PFS.

The literature does not consist of study on MSM operators for the spherical fuzzy environment to date. Therefore, it makes sense to focus on this topic. This concept serves as our inspiration as we concentrate on expanding the MSM to combined SF data and their use in MADM in this study. For this, the remaining of this essay is planned as upcoming. The following chapter provided certain fundamental SFS principles as well as several SFV operating laws. The SFMSM operator is created in Section 3. The SFWMSM operator is created in Section 4. We used the SFWMSM operator in Section 5 to create a MADM ideal with spherical fuzzy material. In Section 6, a real-world illustration is given to approve the established method, show its applicability and efficiency, and provide a comprehensive analysis. Section 7 provides the paper's conclusion and certain closing thoughts.

2. Preliminaries

This chapter discusses some fundamental concepts and terms associated with the planned study. In this chapter, we describe all the words and concepts related to FS, IFS, PyFS, q-ROPFS, PFS, and SFS.

The main concept of SFSs (Mahmood et al., 2019) is detailed reviewed in this chapter. Later, we discuss score and accuracy functions for spherical fuzzy values (SFVs). Moreover, a new assessment technique for SFVs is established.

Definition 1. (Zadeh, 1965)

Assume that X is a non-empty set. An FS is defined as $S = (\omega(x) : x \in X)$ where $\omega : X \to [0,1]$ is known as MD.

Definition 2. (Atanassov, 1986)

Suppose that X is taken non-empty set. An IFS defined as $S = \{(\omega_s(x), \psi_s(x)) : x \in X\}$ such as $\omega_s : X \to [0,1]$ describes MD and $\psi_s : X \to [0,1]$ describes NMD of the element $x \epsilon X$ to S, respectively, and satisfies $0 \le \omega_s(x) + \psi_s(x) \le 1$. The formula for hesitation degree is $r(x) = 1-(\omega_s(x) + \psi_s(x))$, and the pair $(\omega_s(x), \psi_s(x))$ is known as IFV.

Definition 3. (Yager, 2013; Yager 2016)

Suppose that X is taken as non-empty set. A PyFS defined as $S = \{(\omega_s(x), \psi_s(x)) : x \in X\}$ such as $\omega_s : X \to [0,1]$ describes MD and $\psi_s : X \to [0,1]$ describes NMD of the element $x \epsilon X$ to S, respectively, and satisfies condition $0 \le (\omega_s(x))^2 + (\psi_s(x))^2 \le 1$. The hesitancy degree is defined by $r(x) = \sqrt{1-(\omega_s(x))^2 + (\psi_s(x))^2}$, and the duplet $(\omega_s(x), \psi_s(x))$ is known as PyFV.

Definition 4. (Maclaurin, 1729)

Suppose that X is taken as non-empty set. A q-ROPFS defined as $S = \{(\omega_s(x), \psi_s(x)) : x \in X\}$ such as $\omega_s : X \to [0,1]$ describes MD and $\psi_s : X \to [0,1]$ describes the NMD of the element $x \epsilon X$ to S, respectively, and satisfies condition $0 \le (\omega_s(x))^q + (\psi_s(x))^q \le 1$ for q. The hesitancy degree is defined by $r(x) = \sqrt[q]{1-(\omega_s(x))^q + (\psi_s(x))^q}$, and the duplet $(\omega_s(x), \psi_s(x))$ is known as q-ROPFV.

Definition 5. (Cuong & Kreinovich, 2014)

Suppose that X is taken as non-empty set. A PFS is defined as $S = \{(\omega_s(x), \eta_s(x), \psi_s(x)) : x \epsilon X\}$

where $\omega_s : X \to [0,1]$ describes the MD, $\eta_s : X \to [0,1]$ describes the AD, and $\psi_s : X \to [0,1]$ describes the NMD of the element $x \epsilon X$ to S, respectively, and for each $x \epsilon X$, it satisfies the condition $0 \le \omega_s(x) + \eta_s(x) + \psi_s(x) \le 1$. The RD is defined as $r(x) = 1-(\omega_s(x) + \eta_s(x) + \psi_s(x))$, and the triplet $(\omega_s(x), \eta_s(x), \psi_s(x))$ is known as PFV.

Definition 6. (Mahmood et al., 2019)

Suppose that X is a non-empty set. The definition of SFS as

$$S = \{(\omega_s(x), \eta_s(x), \psi_s(x)) : x \epsilon X\}$$

where $\omega_s : X \to [0,1]$ describes the MD, the $\eta_s : X \to [0,1]$ describes the AD, and $\psi_s : X \to [0,1]$ describes the NMD of the element $x \epsilon X$ to S, respectively, and for each $x \in X$, it satisfies that

$$0 \le (\omega_s(x))^2 + (\eta_s(x))^2 + (\psi_s(x))^2 \le 1.$$

The RD is defined as $r(x) = 1-(\omega_s(x) + \eta_s(x) + \psi_s(x))$, and the triplet $(\omega_s(x), \eta_s(x), \psi_s(x))$ is known as SFV. Let $\tilde{\alpha} = (\omega_s(x), \eta_s(x), \psi_s(x))$ be an SFV, $S(\tilde{\alpha}) = \omega^2 - \psi^2 r^2$ where $r = 1-(\omega_s(x) + \eta_s(x) + \psi_s(x))$ and $H(\tilde{\alpha}) = (\omega_s(x))^2 + (\eta_s(x))^2 + (\psi_s(x))^2$ are corresponding score and accuracy function of the SFV $\tilde{\alpha}$.

Definition 7. (Mahmood et al., 2019)

Suppose that $\tilde{\alpha}_1 = (\omega_1, \eta_1, \psi_1)$ and $\tilde{\alpha}_2 = (\omega_2, \eta_2, \psi_2)$ be two SFVs, $S(\tilde{\alpha}_1) = {\omega_1}^2 - {\psi_1}^2{r_1}^2$ and $S(\tilde{\alpha}_2) = {\omega_2}^2 - {\psi_2}^2{r_2}^2$ be the scores of α *and* β, respectively, and let $H(\tilde{\alpha}_1) = {\omega_1}^2 + {\eta_1}^2 + {\psi_1}^2$ and $H(\tilde{\alpha}_2) = {\omega_2}^2 + {\eta_2}^2 + {\psi_2}^2$ be the accuracy degree of α and β correspondingly, then $S(\tilde{\alpha}) < S(\tilde{\beta})$, then $\tilde{\alpha} < \tilde{\beta}$; if $S(\tilde{\alpha}) = S(\tilde{\beta})$,

If $H(\tilde{\alpha}) = H(\tilde{\beta})$, then $\tilde{\alpha} = \tilde{\beta}$; (2) if $H(\tilde{\alpha}) < H(\tilde{\beta})$, then $\tilde{\alpha} < \tilde{\beta}$.

Definition 8. (Mahmood et al., 2019)

Let $\tilde{\alpha}_1 = (\omega_1, \eta_1, \psi_1)$ and $\tilde{\alpha}_2 = (\omega_2, \eta_2, \psi_2)$ and $\tilde{\alpha} = (\omega, \eta, \psi)$ be three SFVs and we apply an elementary operation on them as below:

$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left(\sqrt{(\omega_1)^2 + (\omega_2)^2 - (\omega_1)^2(\omega_2)^2}, \eta_1\eta_2, \psi_1\psi_2\right) \quad (1)$$

$$\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \left(\begin{array}{c} c\omega_1\omega_2, \sqrt{(\eta_1)^2 + (\eta_2)^2 - (\eta_1)^2(\eta_2)^2}, \\ \sqrt{(\psi_1)^2 + (\psi_2)^2 - (\psi_1)^2(\psi_2)^2} \end{array}\right) \quad (2)$$

$$\lambda\tilde{\alpha} = \left(\sqrt{1 - (1 - \omega^2)^\lambda}, \eta^\lambda, \psi^\lambda\right), \quad \lambda = 0; \quad (3)$$

$$(\tilde{\alpha})^\lambda = \left(\omega^\lambda, \sqrt{1 - (1 - \eta^2)^\lambda}, \sqrt{1 - (1 - \psi^2)^\lambda}\right), \quad \lambda > 0; \quad (4)$$

$$\tilde{\alpha}^c = (\psi, \eta, \omega) \quad (5)$$

This technique was first presented by Maclaurin (1729), which is a helpful method categorized by the ability to lock up the affiliation between the involved opinions. The classification of this method is described below:

Definition 9. Suppose that $\alpha_q (q = 1, 2, 3 \cdots, n)$ be a collection of non-negative values, $t = 1, 2, 3 \cdots, n$. Then,

$$MSM^{(t)}(\alpha_1, \alpha_2, \cdots .\alpha_n) = \left(\frac{\sum_{1 \le i_1 \le \ldots \le i_t \le n} \prod_{q=1}^{t} \alpha_{i_q}}{c_n^t}\right)^{\frac{1}{t}}$$

is known as the MSM operator, where $(i_1, i_2, \cdots .i_n)$ traversal entire k-tuple mixture of $(1, 2, \cdots \infty, n)$ and c_n^t is defined as a binomial coefficient. $MSM^{(t)}$ have properties:

$$MSM^{(t)}(0, 0, \cdots, 0) = 0;$$

$$MSM^{(t)}(\alpha, \alpha, \cdots, \alpha) = \alpha;$$

$$MSM^{(t)}(\alpha_1, \alpha_2, \cdots .\alpha_n) \le MSM^{(t)}(\beta_1, \beta_2, \cdots .\beta_n), if \alpha_i \le \beta_i \; for \; all \; i;$$

$$\min_q\{\alpha_i\} \le MSM^{(t)}(\alpha_1, \alpha_2, \cdots .\alpha_n) \le \max_q$$

3. New MSM Based on SFS

Here we define some SFMSM operators based on the process of SFVs.

Definition 10. Let $\alpha_q = (\omega_q, \eta_q, \psi_q) \quad (q = 1, 2, 3 \cdots, n)$ be a collection of SFVs, and then, we define the SFMSM operators as follows:

$$\mathrm{SFMSM}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\frac{1 \le i_1 \le \overset{\oplus}{\ldots} \le i_t \le n(\overset{t}{\underset{q=1}{\otimes}} \tilde{\alpha}_{i_q})}{c_n^t}\right)^{\frac{1}{t}}$$

By getting the information about the SF ideals, we can develop Theorem 1.

Theorem 1. Assume that $\tilde{\alpha}_i = \left(\omega_q, \eta_q, \psi_q\right)(q = 1, 2, \ldots, n)$ be a collection of SFVs, then we describe the SFMSM operator as below:

$$SFMSM(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\begin{array}{c} \left(\sqrt{1 - (\prod_{1 \le i_1 \le \ldots \le i_t \le n}(1 - (\prod_{q=1}^t \omega_{i_q})^2))^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}} \\ \sqrt{1 - \left(1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\left(1 - \prod_{q=1}^t\left(1 - \left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}} \\ \sqrt{1 - \left(1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\left(1 - \prod_{q=1}^t\left(1 - \left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}} \end{array}\right)$$

Proof: By the operating laws of the SFVs, we have

$$\overset{t}{\underset{q=1}{\otimes}} \tilde{\alpha}_{i_q} = \left(\begin{array}{c} \prod_{q=1}^t \omega_{i_q}, \sqrt{1 - \prod_{q=1}^t\left(1 - \left(\eta_{i_q}\right)^2\right)}, \\ \sqrt{1 - \prod_{q=1}^t\left(1 - \left(\psi_{i_q}\right)^2\right)}, \end{array}\right)$$

And

$$1 \le i_1 \le \overset{\oplus}{\ldots} \le i_t \le n\left(\overset{t}{\underset{q=1}{\otimes}} \tilde{\alpha}_{i_q}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \prod_{1 \le i_1 \le \ldots \le i_t \le n}\left(1 - \left(\prod_{q=1}^t \omega_{i_q}\right)^2\right)} \\ \prod_{1 \le i_1 \le \ldots \le i_t \le n}\sqrt{1 - \prod_{q=1}^t\left(1 - \left(\eta_{i_q}\right)^2\right)}, \\ \prod_{1 \le i_1 \le \ldots \le i_t \le n}\sqrt{1 - \prod_{q=1}^t\left(1 - \left(\psi_{i_q}\right)^2\right)} \end{array}\right)$$

Then, we find

$$\frac{1}{C_n^t} 1 \le i_1 \le \overset{\oplus}{\ldots} \le i_t \le n\left(\overset{t}{\underset{q=1}{\otimes}} \tilde{\alpha}_{i_q}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\left(1 - \left(\prod_{q=1}^t \omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^t}}}, \\ \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\sqrt{1 - \prod_{q=1}^t\left(1 - \left(\eta_{i_q}\right)^2\right)}\right)^{\frac{1}{C_n^t}}, \\ \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\sqrt{1 - \prod_{q=1}^t\left(1 - \left(\psi_{i_q}\right)^2\right)}\right)^{\frac{1}{C_n^t}} \end{array}\right)$$

$$\mathrm{SFMSM}^{(t)}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\begin{array}{c} \left(\sqrt{1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\left(1 - \left(\prod_{q=1}^t \omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\sqrt{1 - \prod_{q=1}^t\left(1 - \left(\eta_{i_q}\right)^2\right)}\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\sqrt{1 - \prod_{q=1}^t\left(1 - \left(\psi_{i_q}\right)^2\right)}\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}} \end{array}\right)$$

$$= \left(\begin{array}{c} \left(\sqrt{1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\left(1 - \left(\prod_{q=1}^t \omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\left(1 - \prod_{q=1}^t\left(1 - \left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}} \\ \sqrt{1 - \left(1 - \left(\prod_{1 \le i_1 \le \ldots \le i_t \le n}\left(1 - \prod_{q=1}^t\left(1 - \left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}} \end{array}\right)$$

It is very easy to show that the SFMSM operator has properties:

Property 1. (Idempotency). If all $\tilde{\alpha}_q(q = 1, 2, 3, \cdots, n)$ are equal, i. e. $\tilde{\alpha}_q = \tilde{\alpha}$, $\forall q$, then SFMSM $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \tilde{\alpha}$.

Proof: Since $\tilde{a} = (\omega_{\tilde{a}}, \eta_{\tilde{a}}, \psi_{\tilde{a}})$ based on Theorem 1, we have

$$SFMSM^t(\tilde{\alpha}, \tilde{\alpha}, \cdots, \tilde{\alpha}) = \begin{pmatrix} \left(\sqrt{1 - \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1 - \left(\prod_{q=1}^{t}\omega_{\tilde{a}}\right)^2\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\sqrt{1 - \prod_{q=1}^{t}(1 - (\eta_{\tilde{a}})^2)}\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\sqrt{1 - \prod_{q=1}^{t}(1 - (\psi_{\tilde{a}})^2)}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{t}}} \end{pmatrix} = \begin{pmatrix} \left(1 - \sqrt{\prod_{1\le i_1\le\ldots\le i_t\le n}(1 - (\omega_{\tilde{a}})^{2t})^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1\le i_1\le\ldots\le i_t\le n}(1 - (1 - (\eta_{\tilde{a}})^2)^t)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1\le i_1\le\ldots\le i_t\le n}(1 - (1 - (\psi_{\tilde{a}})^2)^t)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}, \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1 - \left(\left(1 - (\omega_{\sim a})^{2k}\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}, \\ \sqrt{1 - \left(1 - \left(\left(1 - (1 - (\eta_{\tilde{a}})^2)^k\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}, \\ \sqrt{1 - \left(1 - \left(\left(1 - (1 - (\psi_{\tilde{a}})^2)^k\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}, \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1 - (1 - (\omega_{\sim a})^{2t})}\right)^{\frac{1}{t}}, \\ \sqrt{1 - (1 - (1 - (1 - (\eta_{\tilde{a}})^2)^t))^{\frac{1}{t}}}, \\ \sqrt{1 - (1 - (1 - (1 - (\psi_{\tilde{a}})^2)^t))^{\frac{1}{t}}}, \end{pmatrix}$$

$$= (\omega_{\tilde{a}}, \eta_{\tilde{a}}, \psi_{\tilde{a}})$$

The proof is completed.

Property 2. (Commutativity). Let $\tilde{a}_q(q = 1, 2, 3, \cdots, n)$ be a collection of SFVs, $\tilde{a'}_q(q = 1, 2, 3, \cdots, n)$ be any arrangement of $\tilde{a}'_q(q = 1, 2, 3, \cdots, n)$, then

$$SFMSM^t(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = SFMSM^{(q)}(\tilde{\alpha}'_1, \tilde{\alpha}'_2, \cdots, \tilde{\alpha}'_n)$$

Proof: Since $\alpha'_q(q = 1, 2, 3, \cdots, n)$ is any permutation of $\tilde{\alpha}_q$ $(q = 1, 2, 3, \cdots, n)$, based on the definition of SFMSM in equation (5), we have

$$SFMSM^t(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\frac{1 \le i_1 \le \overset{\oplus}{\ldots} \le i_t \le n\left(\underset{q=1}{\overset{t}{\otimes}}\tilde{\alpha}_{i_q}\right)}{C_n^t}\right)^{\frac{1}{t}}$$

$$= \left(\frac{1 \le i_1 \le \overset{\oplus}{\ldots} \le i_t \le n\left(\underset{q=1}{\overset{t}{\otimes}}\tilde{\alpha}_{i_j}\right)}{C_n^t}\right)^{\frac{1}{t}} = SFMSM^{(t)}(\tilde{\alpha}'_1, \tilde{\alpha}'_2, \cdots, \tilde{\alpha}'_n)$$

So, it is complete.

Property 3. (Monotonicity). Let $\tilde{\alpha}_q, \tilde{\alpha}'_q(q = 1, 2, 3, \cdots, n)$ be two collections of SFVs, If $\omega_q \ge \omega'_q$, $\eta_q \le \eta'_q, \psi_q \le \psi'_q$, for all q, then

$$SFMSM^t(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \ge SFMSM^{(t)}(\tilde{\alpha}'_1, \tilde{\alpha}'_2, \cdots, \tilde{\alpha}'_n)$$

Proof: Since $k \ge 1, \omega_q \ge \omega'_q$, $\psi_q \le \psi'_q, \psi_q \le \psi'_q$, then we have $\omega_q \ge \omega'_q \ge 0$, $\psi_q \le \psi'_q \le 0$, $\psi_q \le \psi'_q \le 0$. By the supposition, $\forall$I, $q(i = 1, 2, \cdots, n; j = 1, 2, \cdots, k)$, then we find

$$\prod_{q=1}^{t}\omega_{i_q} \ge \prod_{q=1}^{t}\omega'_{i_q} \Rightarrow 1 - \left(\prod_{q=1}^{t}\omega_{i_q}\right)^2 \le 1 - \left(\prod_{q=1}^{t}\omega'_{i_q}\right)^2$$

$$\Rightarrow \prod_{1\le i_1\le\ldots\le i_k\le n}\left(1 - \left(\prod_{q=1}^{t}\omega_{i_q}\right)^2\right) \le \prod_{1\le i_1\le\ldots\le i_k\le n}\left(1 - \left(\prod_{q=1}^{t}\omega'_{i_q}\right)^2\right)$$

$$\Rightarrow \left(\left(\sqrt{1 - \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1 - \left(\prod_{q=1}^{t}\omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}\right)$$

$$\ge \left(\left(\sqrt{1 - \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1 - \left(\prod_{q=1}^{t}\omega'_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}\right)$$

Likewise, we have

$$\eta_{i_q} \le \eta'_{i_q} \Rightarrow 1 - \left(\eta_{i_q}\right)^2 \le 1 - \left(\eta'_{i_q}\right)^2$$

$$1 - \prod_{q=1}^{t}\left(\eta_{i_q}\right)^2 \le 1 - \prod_{q=1}^{t}\left(\eta'_{i_q}\right)^2$$

$$\Rightarrow \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}} \le \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\eta'_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}$$

$$\Rightarrow \left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}$$

$$\le \left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\eta'_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}$$

$$\Rightarrow \sqrt{\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}$$

$$\le \sqrt{\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\eta'_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}$$

And

$$\psi_{i_q} \le \psi'_{i_q} \Rightarrow 1-\left(\psi_{i_q}\right)^2 \le 1-\left(\psi'_{i_q}\right)^2$$

$$\Rightarrow 1-\prod_{q=1}^{t}\left(\psi_{i_q}\right)^2 \le 1-\prod_{q=1}^{t}\left(\psi'_{i_q}\right)^2$$

$$\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}$$

$$\le \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\psi'_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}$$

$$\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}$$

$$\le \left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\psi'_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}$$

$$\sqrt{\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}$$

$$\le \sqrt{\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\psi'_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}$$

Property 4. (Boundedness). Let $\sim\alpha_q$ $(q=1,2,3,\cdots,n)$ be a collection of SFVs, and let $\omega_q \ge \omega'_q,\ \eta_q \le \eta'_q, \psi_q \le \psi'_q,$ for all q, and

$$\tilde{\alpha}^- = \min_q \tilde{\alpha}_q = \left(\min_q {}_{\omega_q,}\ \max_q {}_{\eta_q,}\ \max_q {}_{\psi_q,}\right)$$

$$\tilde{\alpha}^+ = \max_q \tilde{\alpha}_q = \left(\max_q {}_{\omega_q,}\ \min_q {}_{\eta_q,}\ \min_q {}_{\psi_q,}\right)$$

Then,

$$\tilde{\alpha}^- \le SFMSM^{(t)}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n) \le \tilde{\alpha}^+$$

Proof: From the property 1 and 3, we have

$$SFMSM^{(t)}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n) \ge SFMSM^{(t)}(\tilde{\alpha}^-,\tilde{\alpha}^-,\cdots,\tilde{\alpha}^-) = \tilde{\alpha}^-$$

$$SFMSM^{(t)}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n) \le SFMSM^{(t)}(\tilde{\alpha}^+,\tilde{\alpha}^+,\cdots,\tilde{\alpha}^+) = \tilde{\alpha}^+$$

Thus, the proof is completed.

Lemma 1. (Maclaurin inequality). Let $\alpha_i(i=1,2,\cdots,n)$ is a set with positive real values and for$k=1,2,\cdots,n$. So,

$MSM^{(1)}(\alpha_1,\alpha_2,\cdots.\alpha_n) \ge MSM^{(2)}(\alpha_1,\alpha_2,\cdots.\alpha_n) \ge \cdots \ge MSM^{(n)}(\alpha_1,\alpha_2,\cdots.\alpha_n)$ with equivalence if and only if $\alpha_1=\alpha_2=\cdots=\alpha_n$

Lemma 2. Let $\alpha_t>0, \beta_t>0(t=1,2,\cdots,m)$, and $\sum_{t=1}^{m}\ \beta_t=1$, then

$$\prod_{t=1}^{m}(\alpha_t)^{\beta_t} \le \sum_{t=1}^{m}\alpha_t\beta_t$$

With equality if and only if $\alpha_1=\alpha_2=\cdots=m$.

Proof: Its proof is simple.

Theorem 2. Given that the SFVs, $\tilde{\alpha}_i(i=1,2,3,\cdots,n)$ and $t=1,2,\cdots,n$, the SFMSM is said to be monotonically decreased if it depends upon parameter t.

Proof. From Theorem 3.1.2, we can have

$$SFMSM^{(t)}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n) = \begin{pmatrix} \left(\sqrt{1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\left(\prod_{q=1}^{t}\omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}, \\ \sqrt{1-\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}, \\ \sqrt{1-\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}} \end{pmatrix}$$

Let

$$f(t) = \left(\sqrt{1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\left(\prod_{q=1}^{t}\omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}$$

$$h(t) = \sqrt{1-\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}$$

$$g(t) = \sqrt{1-\left(1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\prod_{q=1}^{t}\left(1-\left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}}$$

Firstly, we will prove $f(t)$ is monotonically decreasing with the parameter t. From Lemma 1 and 2, we can have

$$f(t) = \left(\sqrt{1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\left(\prod_{q=1}^{t}\omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}}$$

$$\ge \left(\sqrt{1-\sum_{1\le i_1\le\ldots\le i_t\le n}\left(1-\left(\prod_{q=1}^{t}\omega_{i_q}\right)^2\right)C_n^t}\right)^{\frac{1}{t}}$$

$$= \left(1-\left(1-\sum_{1\le i_1\le\ldots\le i_t\le n}\left(\prod_{q=1}^{t}\omega_{i_q}\right)^2 C_n^t\right)\right)^{\frac{1}{t}}$$

$$= \left(\sum_{1\le i_1\le\ldots\le i_t\le n}\left(\prod_{q=1}^{t}\omega_{i_q}\right)^2 C_n^t\right)^{\frac{1}{t}}$$

After that, we will prove it by the contradiction method. Assume that $f(t)$ is monotonically increasing. So, $f(n)>f(n+1)>\cdots>f(1)$ Also, since $f(1) \ge \left(\sum_{1\le i_1\le\ldots\le i_t\le n}\left(\prod_{q=1}^{1}\omega_{i_q}\right)^2 C_n^1\right)^{11} = \sum_{i=1}^{n}\omega_i^2 n$. Then, it follows that $(n)>f(1) \ge \left(\sum_{1\le i_1\le\ldots\le i_t\le n}\ \left(\prod_{q=1}^{1}\omega_{i_j}\right)^2 C_n^1\right)^{11} = \sum_{i=1}^{n}\omega_i^2 n \Rightarrow \left(\prod_{i=1}^{n}\omega_{\mathrm{i}}^2\right)^{\frac{1}{n}} > \sum_{i=1}^{n}\omega_i^2 n$. Now by the elementary mean inequality (Pecaric et al., 2005), $\left(\prod_{i=1}^{n}\omega_i^2\right)^{\frac{1}{n}} \leq \sum_{i=1}^{n}\omega_i^2 n$. It is very easy to see that it contradicts inequality. So, $f(t)$ is monotonically decreasing. Likewise, $h(t)$ and $g(t)$ can also be proved monotonically increasing.

From the score purpose of SFVs,

$$s(t) = (f^2(t) - g^2(t)r^2(t))$$

For any $t \in [1, n]$, and $t \in Z$, we can get

$$s(t+1) - s(t)$$
$$= [(f^2(t+1) - g^2(t+1)r^2(t+1)) - (f^2(t) - g^2(t)r^2(t))]$$
$$= [(f^2(t+1) - f^2(t)) + (g^2(t)r^2(t) - g^2(t+1)r^2(t+1))]$$
$$= \begin{bmatrix} (f(t+1) - f(t))(f(t+1) + f(t)) \\ +(g(t)r(t) - g(t+1)r(t+1))(g(t)r(t) + g(t+1)r(t+1)) \end{bmatrix}$$

Since the function is monotonically decreasing with the respect to the parameter t, so$(f(t+1) - f(t)) < 0$; the function $g(t)$ is monotonically increasing with the respect to the parameter t, so we can imply $(g(k)r(t) - g(k+1)r(t+1)) < 0$. Based on the above result, we have

$$s(k+1) - s(k)$$
$$= \begin{bmatrix} (f(t+1) - f(t))(f(t+1) + f(t)) \\ +(g(t)r(t) - g(t+1)r(t+1)) \\ (g(t)r(t) + g(t+1)r(t+1)) \end{bmatrix} < 0$$

So, $s(t+1) < s(t)$, $\forall$ t, and the SFMSM are also monotonically decreasing with the respect to the parameter t. Hence, Theorem is completed.

After that, we will talk about some special cases of the SFMSM operator concerning the parameter t.

As $t = 1$, the SFMSM operator reduces to the spherical fuzzy average operator as below:

$$SFMSM^{(1)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) =$$

$$\begin{pmatrix} \left(\sqrt{1 - \left(\prod_{1 \leq i_1 \leq n}\left(1 - \left(\prod_{q=1}^{1}\omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^1}}}\right)^{11}, \\ \sqrt{1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n}\left(1 - \prod_{q=1}^{1}\left(1 - \left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^1}}\right)^{11}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1 \leq i_1 \leq n}\left(1 - \prod_{q=1}^{1}\left(1 - \left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^1}}\right)^{11}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - \left(\prod_{1 \leq i_1 \leq n}\left(1 - (\omega_{i_1})^2\right)\right)^{\frac{1}{n}}}, \\ \sqrt{\left(\prod_{1 \leq i_1 \leq n}\left(1 - \left(1 - (\eta_{i_1})^2\right)\right)\right)^{\frac{1}{n}}}, \\ \sqrt{\left(\prod_{1 \leq i_1 \leq n}\left(1 - \left(1 - (\psi_{i_1})^2\right)\right)\right)^{\frac{1}{n}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - \left(\prod_{1 \leq i_1 \leq n}\left(1 - (\omega_{i_1})^2\right)\right)^{\frac{1}{n}}}, \\ \prod_{1 \leq i_1 \leq n}(\eta_i)^{\frac{1}{n}} \\ \prod_{1 \leq i_1 \leq n}(\psi_i)^{\frac{1}{n}} \end{pmatrix} (let i_1 = i)$$

$$= \begin{pmatrix} \sqrt{1 - \left(\prod_{1 \leq i_1 \leq n}\left(1 - (\omega_{i_1})^2\right)\right)^{\frac{1}{n}}}, \\ \prod_{i=1}^{n}(\eta_i)^{\frac{1}{n}} \\ \prod_{I=1}^{n}(\psi_i)^{\frac{1}{n}} \end{pmatrix}$$

Now we take $t = 2$,

$$SFMSM^{(2)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) =$$

$$\begin{pmatrix} \left(\sqrt{1 - \left(\prod_{1 \leq i_1 \leq I_2 \leq n}\left(1 - \left(\prod_{q=1}^{2}\omega_{i_q}\right)^2\right)\right)^{1C_n^2}}\right)^{\frac{1}{2}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1 \leq i_1 \leq I_2 \leq n}\left(1 - \prod_{q=1}^{2}\left(1 - \left(\eta_{i_q}\right)^2\right)\right)\right)^{1C_n^2}\right)^{\frac{1}{2}}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1 \leq i_1 \leq I_2 \leq n}\left(1 - \prod_{q=1}^{2}\left(1 - \left(\psi_{i_q}\right)^2\right)\right)\right)^{1C_n^2}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1 - \left(\prod_{1 \leq 1 \leq I_2 \leq n}\left(1 - (\omega_{i_1}\omega_{i_2})^2\right)\right)^{\frac{2}{n(n-1)}}}\right), \\ \sqrt{1 - \left(1 - \left(\prod_{1 \leq i_1 \leq I_2 \leq n}\left(1 - \left(1 - (\eta_{i_1})^2\right)\left(1 - (\eta_{i_2})^2\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1 \leq i_1 \leq I_2 \leq n}\left(1 - \left(1 - (\psi_{i_1})^2\right)\left(1 - (\psi_{i_2})^2\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1 - \left(\prod_{i_1, i_2 = 1 i_1 \neq i_2}^{n}\left(1 - (\omega_{i_1}\omega_{i_2})^2\right)\right)^{\frac{1}{22n(n-1)}}}\right) \\ \sqrt{1 - \left(1 - \left(\prod_{i_1, i_2 = 1 i_1 \neq i_2}^{n}\left(1 - \left(1 - (\eta_{i_1})^2\right)\left(1 - (\eta_{i_2})^2\right)\right)\right)^{\frac{\frac{1}{22}}{n(n-1)}}\right)^{\frac{1}{2}}}, \\ \sqrt{1 - \left(1 - \left(\prod_{i_1, i_2 = 1 i_1 \neq i_2}^{n}\left(1 - \left(1 - (\psi_{i_1})^2\right)\left(1 - (\psi_{i_2})^2\right)\right)\right)^{\frac{\frac{1}{22}}{n(n-1)}}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\sqrt{1 - \left(\prod_{i_1, i_2 = 1 i_1 \neq i_2}^{n}\left(1 - (\omega_{i_1}\omega_{i_2})^2\right)\right)^{\frac{2}{n(n-1)}}}\right)^{\frac{1}{2}}, \\ \sqrt{1 - \left(1 - \left(\prod_{i_1, i_2 = 1 i_1 \neq i_2}^{n}\left(1 - \left(1 - (\eta_{i_1})^2\right)\left(1 - (\eta_{i_2})^2\right)\right)\right)^{\frac{2}{n(n-1)}}\right)^{\frac{1}{2}}}, \\ \sqrt{1 - \left(1 - \left(\prod_{i_1, i_2 = 1 i_1 \neq i_2}^{n}\left(1 - \left(1 - (\psi_{i_1})^2\right)\left(1 - (\psi_{i_2})^2\right)\right)\right)^{\frac{\frac{2}{n}}{(n-1)}}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$= SFBM(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$$

When $t = n$, the SFMSM operator decreases to the SF geometric mean operator as below:

$$SFMSM^{(n)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \begin{pmatrix} \left(\sqrt{1 - \left(\prod_{1\le i_1 \le I_2 \le n}\left(1 - \left(\prod_{q=1}^{n} \omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^n}}}\right)^{\frac{1}{2}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1\le i_1 \le I_2 \le n}\left(1 - \prod_{q=1}^{n}\left(1 - \left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{2}}}, \\ \sqrt{1 - \left(1 - \left(\prod_{1\le i_1 \le I_2 \le n}\left(1 - \prod_{q=1}^{n}\left(1 - \left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^n}}\right)^{\frac{1}{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - \left(1 - \left(\prod_{q=1}^{n}\left(\omega_{i_q}\right)^2\right)\right)^{\frac{1}{C_n^n}}} \\ \sqrt{1 - \left(1 - \left(1 - \prod_{q=1}^{n}\left(1 - \left(\eta_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^n}}}^{\frac{1}{2}}, \\ \sqrt{1 - \left(1 - \left(1 - \prod_{q=1}^{n}\left(1 - \left(\psi_{i_q}\right)^2\right)\right)\right)^{\frac{1}{C_n^n}}}^{\frac{1}{2}}, \end{pmatrix} (let i_q = i) = \begin{pmatrix} \left(\prod_{i=1}^{n} \omega_i\right)^{\frac{1}{n}} \\ \sqrt{1 - \left(\prod_{i=1}^{n}\left(1 - (\eta_i)^2\right)\right)^{\frac{1}{n}}} \\ \sqrt{1 - \left(\prod_{i=1}^{n}\left(1 - (\psi_i)^2\right)\right)^{\frac{1}{n}}} \end{pmatrix}$$

Example 1. Let $\tilde{\alpha}_1 = (0.4, 0.7, 0.3), \tilde{\alpha}_2 = (0.5, 0.4, 0.2), \tilde{\alpha}_3 = (0.8, 0.3, 0.5)$, and $\tilde{\alpha}_4 = (0.6, 0.3, 0.4)$ are four SFVs. Now we take the SFMSM operator to evaluate the four SFVs, take $T = 2$,

$$\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \left(0.4 \times 0.5, \sqrt{0.7^2 + 0.4^2 - 0.7^2 \times 0.4^2}, \sqrt{0.3^2 + 0.2^2 - 0.3^2 \times 0.2^2}\right) = (0.20, 0.76, 0.36)$$

$$\tilde{\alpha}_1 \otimes \tilde{\alpha}_3 = \left(0.4 \times 0.8, \sqrt{0.7^2 + 0.3^2 - 0.7^2 \times 0.3^2}, \sqrt{0.3^2 + 0.5^2 - 0.3^2 \times 0.5^2}\right) = (0.32, 0.73, 0.56)$$

$$\tilde{\alpha}_1 \otimes \tilde{\alpha}_4 = \left(0.4 \times 0.6, \sqrt{0.7^2 + 0.3^2 - 0.7^2 \times 0.3^2}, \sqrt{0.3^2 + 0.4^2 - 0.3^2 \times 0.4^2}\right) = (0.24, 0.73, 0.49)$$

$$\tilde{\alpha}_2 \otimes \tilde{\alpha}_3 = \left(0.5 \times 0.8, \sqrt{0.4^2 + 0.3^2 - 0.4^2 \times 0.3^2}, \sqrt{0.2^2 + 0.5^2 - 0.2^2 \times 0.5^2}\right) = (0.40, 0.49, 0.53)$$

$$\tilde{\alpha}_2 \otimes \tilde{\alpha}_4 = \left(0.5 \times 0.6, \sqrt{0.4^2 + 0.3^2 - 0.4^2 \times 0.3^2}, \sqrt{0.2^2 + 0.4^2 - 0.2^2 \times 0.4^2}\right) = (0.30, 0.49, 0.44)$$

$$\tilde{\alpha}_3 \otimes \tilde{\alpha}_4 = \left(0.8 \times 0.6, \sqrt{0.3^2 + 0.3^2 - 0.3^2 \times 0.3^2}, \sqrt{0.5^2 + 0.4^2 - 0.5^2 \times 0.4^2}\right) = (0.48, 0.41, 0.61)$$

By Equation, we get

$$SFMSM^{(2)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4) = \left(c \oplus 1 \le i_1 \le i_2 \le n\left(\tilde{\alpha}_{i_1} \otimes \tilde{\alpha}_{i_2}\right) C_4^2\right)^{12} = (0.58, 0.43, 0.49)$$

4. New Weighted MSM Based on SFS

It is cleared from Section 3 that the SFMSM operator does not take the weight of the combined arguments into account. However, the weights of the features produce a significant part in the progression of aggregation in many real-world circumstances, particularly when making decisions based on various attributes. We will put out the SFWMSM operator in the manner as follows to get over SFMSM's drawback.

Definition 11. Let $\tilde{\alpha}_q = (\omega_q, \eta_q, \psi_q)(q = 1, 2, \ldots, n)$ be a collection of SFVs, $w = (w_1, w_2, \cdots, w_n)$ is a weight vector of $\tilde{\alpha}_q(q = 1, 2, \cdots, n)$, and $w_j > 0$, $\sum_{q=1}^{n} w_q = 1$. If $SFMSM_w{}^{(t)}\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \overset{\oplus}{\ldots}, \tilde{\alpha}_n\right) = \frac{1\le i_1 \le \ldots \le i_t \le n\left(\overset{t}{\underset{q=1}{\otimes}}\left(\tilde{\alpha}_{i_q}\right)^{w_{i_q}}\right)}{C_n^t}$. So, the SFMSM is known as the SFWMSM operator.

By the operations of SFVs established in chapter 2, we also can develop the following Theorem.

Theorem 3. Suppose that $1 \le t \le n (t \in Z)$ and $\tilde{\alpha}_q = (\omega_q, \eta_q, \psi_q)$ $(q = 1, 2, \ldots, n)$ be a collection of SFVs, $w = (w_1, w_2, \cdots, w_n)$ is a weight vector of $\tilde{\alpha}_q(q = 1, 2, \cdots, n)$, and $w_q > 0$, $\sum_{q=1}^{n} w_q = 1$. Then, the grouped value by using the SFWMSM is also an SFV, and

$$SFMSM_w{}^{(t)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \frac{1 \le i_1 \le \overset{\oplus}{\ldots} \le i_t \le n\left(\overset{t}{\underset{q=1}{\otimes}}\left(\tilde{\alpha}_{i_q}\right)^{w_{i_q}}\right)}{C_n^t}$$

$$= \begin{pmatrix} \sqrt{1 - \left(\prod_{1\le i_1 \le \cdots \le i_t \le n}\left(1 - \left(\prod_{q=1}^{t}\left(\omega_{i_q}\right)^{w_{i_q}}\right)^2\right)\right)^{\frac{1}{C_n^t}}} \\ \left(\prod_{1\le i_1 \le \cdots \le i_t \le n}\sqrt{1 - \prod_{q=1}^{t}\left(1 - \left(\eta_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^t}} \\ \left(\prod_{1\le i_1 \le \cdots \le i_t \le n}\sqrt{1 - \prod_{q=1}^{t}\left(1 - \left(\psi_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^t}} \end{pmatrix}$$

Proof: By the working rules of SFVs, we use

$$\overset{t}{\underset{q=1}{\otimes}}\left(\tilde{\alpha}_{i_q}\right)^{w_{i_q}} = \begin{pmatrix} \prod_{q=1}^{t}\left(\omega_{i_q}\right)^{w_{i_q}}, \sqrt{1 - \prod_{q=1}^{t}\left(1 - \left(\eta_{i_q}\right)^2\right)^{w_{i_q}}}, \\ \sqrt{1 - \prod_{q=1}^{t}\left(1 - \left(\psi_{i_q}\right)^2\right)^{w_{i_q}}} \end{pmatrix}$$

And

$$\le i_1 \le \overset{\oplus}{\ldots} \le i_t \le n\left(\underset{q=1}{\overset{t}{\otimes}} \left(\sim\alpha_{i_q}\right)^{w_{i_q}} \right) =$$

$$\begin{pmatrix} \left(\sqrt{1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\left(\prod_{q=1}^{t}\left(\omega_{i_q}\right)^{w_{i_q}}\right)^2\right)\right)}\right) \\ \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\sqrt{1-\prod_{q=1}^{t}\left(1-\left(\mathfrak{y}_{i_q}\right)^2\right)^{w_{i_q}}}\right) \\ \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\sqrt{1-\prod_{q=1}^{t}\left(1-\left(\psi_{i_q}\right)^2\right)^{w_{i_q}}}\right) \end{pmatrix}$$

Then, we find

$$\frac{1}{C_n^t} 1 \le i_1 \le \overset{\oplus}{\ldots} \le i_t \le n\left(\underset{q=1}{\overset{t}{\otimes}} \left(\tilde{\alpha}_{i_q}\right)^{w_{i_q}} \right) =$$

$$\begin{pmatrix} \sqrt{1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\left(\prod_{q=1}^{t}\left(\omega_{i_q}\right)^{w_{i_q}}\right)^2\right)\right)^{\frac{1}{C_n^t}}} \\ \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\sqrt{1-\prod_{q=1}^{t}\left(1-\left(\mathfrak{y}_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^t}} \\ \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\sqrt{1-\prod_{q=1}^{t}\left(1-\left(\psi_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^t}} \end{pmatrix}$$

Therefore,

$$SFMSM_w{}^{(t)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \begin{pmatrix} \sqrt{1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\left(\prod_{q=1}^{t}\left(\omega_{i_q}\right)^{w_{i_q}}\right)^2\right)\right)^{\frac{1}{C_n^t}}} \\ \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\sqrt{1-\prod_{q=1}^{t}\left(1-\left(\mathfrak{y}_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^t}} \\ \left(\prod_{1\le i_1\le\ldots\le i_t\le n}\sqrt{1-\prod_{q=1}^{t}\left(1-\left(\psi_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^t}} \end{pmatrix}$$

Then, we also can debate certain distinct cases of the SFWMSM operator concerning the parameter T.

As $t = 1$, the SFWMSM operator decreases in the following shape:

$$SFMSM_w{}^{(1)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \begin{pmatrix} \sqrt{1-\left(\prod_{1\le i_1\le n}\left(1-\left(\prod_{q=1}^{1}(\omega_{i_1})^{w_{i_1}}\right)^2\right)\right)^{\frac{1}{C_n^1}}} \\ \left(\prod_{1\le i_1\le n}\sqrt{1-\prod_{q=1}^{1}(1-(\mathfrak{y}_{i_1})^2)^{w_{i_1}}}\right)^{\frac{1}{C_n^1}} \\ \left(\prod_{1\le i_1\le n}\sqrt{1-\prod_{q=1}^{1}(1-(\psi_{i_1})^2)^{w_{i_1}}}\right)^{\frac{1}{C_n^1}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-\left(\prod_{1\le i_1\le n}(1-(\omega_{i_1})^{2w_{i_1}})\right)^{\frac{1}{n}}} \\ \left(\prod_{1\le i_1\le n}\sqrt{1-\prod_{j=1}^{1}(1-(\mathfrak{y}_{i_1})^2)^{w_{i_1}}}\right)^{\frac{1}{n}} \\ \left(\prod_{1\le i_1\le n}\sqrt{1-\prod_{j=1}^{1}(1-(\psi_{i_1})^2)^{w_{i_1}}}\right)^{\frac{1}{n}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-\left(\prod_{1\le i_1\le n}(1-(\omega_{i_1})^{2w_{i_1}})\right)^{\frac{1}{n}}}^{\frac{1}{n}} \\ \left(\prod_{1\le i_1\le n}\sqrt{1-(1-(\mathfrak{y}_{i_1})^2)^{w_{i_1}}}\right)^{\frac{1}{n}} \\ \left(\prod_{1\le i_1\le n}\sqrt{1-(1-(\psi_{i_1})^2)^{w_{i_1}}}\right) \end{pmatrix} (let i_1 = q)$$

$$= \begin{pmatrix} \sqrt{1-\left(\prod_{q=1}^{n}(1-(\omega_q)^{2w_q})\right)^{\frac{1}{n}}} \\ \left(\prod_{q=1}^{n}\sqrt{1-\left(1-\left(\mathfrak{y}_q\right)^2\right)^{w_q}}\right)^{\frac{1}{n}} \\ \left(\prod_{q=1}^{n}\sqrt{1-(1-(\psi_q)^2)^{w_q}}\right)^{\frac{1}{n}} \end{pmatrix}$$

When $t = 2$, the SFWMSM operator decreases to the SFBM operator as below:

$$SFMSM^{(2)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \begin{pmatrix} \sqrt{1-\left(\prod_{1\le i_1\le i_2\le n}\left(1-\left(\prod_{q=1}^{2}\left(\omega_{i_q}\right)^{w_{i_q}}\right)^2\right)\right)^{\frac{1}{C_n^2}}} \\ \left(\prod_{1\le i_1\le i_2\le n}\sqrt{1-\prod_{q=1}^{2}\left(1-\left(\mathfrak{y}_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^2}} \\ \left(\prod_{1\le i_1\le i_2\le n}\sqrt{1-\prod_{q=1}^{2}\left(1-\left(\psi_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-\left(\prod_{1\le i_1\le i_2\le n}(1-((\omega_{i_1})^{w_{i_1}}(\omega_{i_2})^{w_{i_2}})^2)\right)^{\frac{2}{n(n-1)}}} \\ \left(\prod_{1\le i_1\le i_2\le n}\sqrt{1-(1-(\mathfrak{y}_{i_1})^2)^{w_{i_1}}(1-(\mathfrak{y}_{i_2})^2)^{w_{i_2}}}\right)^{\frac{2}{n(n-1)}} \\ \left(\prod_{1\le i_1\le i_2\le n}\sqrt{1-(1-(\psi_{i_1})^2)^{w_{i_1}}(1-(\psi_{i_2})^2)^{w_{i_2}}}\right)^{\frac{2}{n(n-1)}} \end{pmatrix}$$

$$= SFMSM(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$$

When $t = n$, SFMSM decreases to the SFWGM operator as follows:

$$SFWMSM_w{}^{(n)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \frac{1 \le i_1 \le \overset{\oplus}{\ldots} \le i_n \le n\left(\underset{q=1}{\overset{t}{\otimes}} \left(\tilde{\alpha}_{i_q}\right)^{w_{i_q}} \right)}{C_n^n}$$

$$= \begin{pmatrix} \sqrt{1-\left(\prod_{1\le i_1\le\cdots\le i_n\le n}\left(1-\left(\prod_{q=1}^{n}\left(\omega_{i_q}\right)^{w_{i_q}}\right)^2\right)\right)^{\frac{1}{C_n^n}}} \\ \left(\prod_{1\le i_1\le\cdots\le i_n\le n}\sqrt{1-\prod_{q=1}^{n}\left(1-\left(\eta_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^n}} \\ \left(\prod_{1\le i_1\le\cdots\le i_n\le n}\sqrt{1-\prod_{q=1}^{n}\left(1-\left(\psi_{i_q}\right)^2\right)^{w_{i_q}}}\right)^{\frac{1}{C_n^n}} \end{pmatrix}$$

$$= \begin{pmatrix} \prod_{q=1}^{n}\left(\omega_{i_q}\right)^{w_{i_q}}, \sqrt{1-\prod_{q=1}^{n}\left(1-\left(\eta_{i_q}\right)^2\right)^{w_{i_q}}}, \\ \sqrt{1-\prod_{q=1}^{n}\left(1-\left(\psi_{i_q}\right)^2\right)^{w_{i_q}}} \end{pmatrix}$$

$$\left(let i_q = q\right)$$

$$= \begin{pmatrix} \prod_{q=1}^{n}(\omega_q)^{w_q}, \sqrt{1-\prod_{q=1}^{n}\left(1-(\eta_q)^2\right)^{w_q}}, \\ \sqrt{1-\prod_{q=1}^{n}\left(1-(\psi_q)^2\right)^{w_q}} \end{pmatrix}$$

Example 2. Let $\tilde{\alpha}_1 = (0.4, 0.7, 0.3), \tilde{\alpha}_2 = (0.5, 0.4, 0.2), \tilde{\alpha}_3 = (0.2, 0.3, 0.5)$, and $\tilde{\alpha}_4 = (0.6, 0.3, 0.4)$ be four SFVs, $w = (0.2, 0.1, 0.3, 0.4)$ is the weight of $\tilde{\alpha}_i (i = 1, 2, 3, 4)$. Now we apply SFWMSM to find the four SFVs. Without any hesitation, Take $t = 2$,

$$(\tilde{\alpha}_1)^{0.2} \otimes (\tilde{\alpha}_2)^{0.1} = \begin{pmatrix} 0.4^{0.2} \times 0.5^{0.1}, \sqrt{1-(1-0.7^2)^{0.2} \times (1-0.4^2)^{0.1}}, \\ \sqrt{1-(1-0.3^2)^{0.2} \times (1-0.2^2)^{0.1}} \end{pmatrix} = (0.78, 0.38, 0.15)$$

$$(\tilde{\alpha}_1)^{0.2} \otimes (\tilde{\alpha}_3)^{0.3} = \begin{pmatrix} 0.4^{0.2} \times 0.2^{0.1}, \sqrt{1-(1-0.7^2)^{0.2} \times (1-0.3^2)^{0.3}}, \\ \sqrt{1-(1-0.3^2)^{0.2} \times (1-0.5^2)^{0.3}} \end{pmatrix} = (0.51, 0.39, 0.32)$$

$$(\tilde{\alpha}_1)^{0.2} \otimes (\tilde{\alpha}_4)^{0.2} = \begin{pmatrix} 0.4^{0.2} \times 0.6^{0.4}, \sqrt{1-(1-0.7^2)^{0.2} \times (1-0.3^2)^{0.4}}, \\ \sqrt{1-(1-0.3^2)^{0.2} \times (1-0.4^2)^{0.4}} \end{pmatrix} = (0.68, 0.40, 0.29)$$

$$(\tilde{\alpha}_2)^{0.1} \otimes (\tilde{\alpha}_3)^{0.3} = \begin{pmatrix} 0.5^{0.1} \times 0.2^{0.3}, \sqrt{1-(1-0.4^2)^{0.1} \times (1-0.3^2)^{0.3}}, \\ \sqrt{1-(1-0.2^2)^{0.1} \times (1-0.5^2)^{0.3}} \end{pmatrix} = (0.58, 0.21, 0.29)$$

$$(\tilde{\alpha}_2)^{0.1} \otimes (\tilde{\alpha}_4)^{0.4} = \begin{pmatrix} 0.5^{0.1} \times 0.6^{0.4}, \sqrt{1-(1-0.4^2)^{0.1} \times (1-0.3^2)^{0.4}}, \\ \sqrt{1-(1-0.2^2)^{0.1} \times (1-0.4^2)^{0.4}} \end{pmatrix} = (0.76, 0.23, 0.27)$$

$$(\tilde{\alpha}_3)^{0.3} \otimes (\tilde{\alpha}_4)^{0.4} = \begin{pmatrix} 0.2^{0.3} \times 0.6^{0.4}, \sqrt{1-(1-0.3^2)^{0.3} \times (1-0.3^2)^{0.4}}, \\ \sqrt{1-(1-0.5^2)^{0.3} \times (1-0.4^2)^{0.4}} \end{pmatrix} = (0.50, 0.25, 0.38)$$

We get

$$SFMS^{(2)}(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4) = \frac{1 \le i_1 \le \overset{\oplus}{\dots} \le i_t \le n\left(\left(\tilde{\alpha}_{i_1}\right)^{w_{i_1}} \otimes \left(\tilde{\alpha}_{i_2}\right)^{w_{i_2}}\right)}{C_4^2} = (0.81, 0.21, 0.25)$$

5. Model for DM with Multiple Attributes Using Spherical Fuzzy Information

We will use the SFWMSM operator to MADM with SF material. The subsequent norms are used to symbolize MADM problems for possible assessment of clothes with SF material. Assume that is a set of alternatives and $G = \{G_1, G_2, \cdots, G_n\}$ be the set of attributes. Let $w = (w_1, w_2, \cdots, w_n)^k$ is taken as a weight vector of attributes, and $w_q \in [0, 1], q = 1, 2, \cdots, n, \sum_{j=1}^{n} w_q = 1$. Assume as $\tilde{E} = \left(\tilde{e}_{iq}\right)_{m\times n} = \left(\omega_{i_q}, \mathfrak{y}_{i_q}, \psi_{i_q}\right)_{m\times n}$ is the SF information matrix, and ω_{i_q} shows the degree of alternative satisfies attribute G_q assumed by the choice creator, $\mathfrak{y}_{i_q}$ shows abstinence behavior of alternative about attribute G_q, ψ_{i_q} shows the degree of alternative satisfies the attribute G_q assumed by the choice creator, $\omega_{i_q} \in [0, 1], \eta_{i_q} \in [0, 1], \mathfrak{y}_{i_q} \in [0, 1]$, $\left(\omega_{i_q}\right)^2 + \left(\eta_{i_q}\right)^2 + \left(\psi_{i_q}\right)^2 \le 1$, $i = 1, 2, \cdots, n, q = 1, 2, \dots, n$.

In the following, we put on the SFWMSM operator to the MADM problems for possible assessment of clothes with SF information.

Step 1: Utilize the data from matrix E and the SFWMSM operators

$$\tilde{e}_i = (\omega_i, \mathfrak{y}_i, \psi_i) = SFWMSM_{w}{}^{(k)}(\tilde{e}_{i1}, e, \cdots, \tilde{e}_{in}) = \frac{1 \le i_1 \le \overset{\oplus}{\dots} \le i_t \le n\left(\overset{t}{\underset{s=1}{\otimes}}\left(\tilde{r}_{i_{js}}\right)^{w_{i_{js}}}\right)^{\frac{1}{t}}}{C_n^t}$$

$$= \begin{pmatrix} \left(\sqrt{1-\left(\prod_{1\le i_1\le\ldots\le i_t\le n}\left(1-\left(\prod_{=1}^{t}\omega_{i_{qs}}\right)^{2w_{qs}}\right)\right)^{\frac{1}{C_n^t}}}\right)^{\frac{1}{t}} \\ \sqrt{1-\left(1-\left(\prod_{1\le i_1\le I_2\le n}\left(1-\prod_{s=1}^{t}\left(1-\left(\eta_{i_{qs}}\right)^2\right)^{w_{qs}}\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}} \\ \sqrt{1-\left(1-\left(\prod_{1\le i_1\le I_2\le n}\left(1-\prod_{s=1}^{t}\left(1-\left(\psi_{i_{qs}}\right)^2\right)^{w_{qs}}\right)\right)^{\frac{1}{C_n^t}}\right)^{\frac{1}{t}}} \end{pmatrix}$$

To arise the whole liking values $\tilde{e}(i = 1, 2, \cdots, m)$ of the attributes

Step 2: Determine scores $S(\tilde{e}_i)(i = 1, 2, \cdots, m)$ of all SF values $\sim e_i(i = 1, 2, \cdots, m)$ for ordering every alternative and find out better one(s). In case we found the same two scores $S(\tilde{e}_i)$ *and* $S(e_q)$, then we will evaluate accuracy degrees $H(\tilde{e}_i)$ *and* $H(\tilde{e}_q)$, correspondingly, and then, we will rank the alternatives according to the accuracy degrees $H(\tilde{e}_i)$ and $\mathrm{H}(\tilde{e}_q)$.

Step 3: Order entire alternatives and pick the finest out of them according to the $S(\tilde{e}_i)(i = 1, 2, \cdots, m)$.

Step 4: End.

6. Illustrative Example and Comparative Analysis

Here we perform a numerical example by using SF data to show the formula offered in this paper. There are five members of the alternatives for selection. The members select four characteristics of clothes (1)G_1 is the thread of clothes, (2)G_2 is the design of clothes, (3)G_3 is the color of clothes, (4)G_4 is normal in range. The five members of the alternatives will be judged by utilizing SF material from the choice maker under the same characteristics which have the weighting vector $w = (0.2, 0.1, 0.3, 0.4)$ as mentioned below.

$$\tilde{E} = \begin{bmatrix} (0.4, 0.3, 0.2), (0.3, 0.5, 0.7), (0.2, 0.3, 0.6), (0.4, 0.2, 0.3) \\ (0.6, 0.3, 0.4), (0.3, 0.5, 0.2), (0.5, 0.4, 0.3), (0.4, 0.6, 0.4) \\ (0.4, 0.2, 0.3), (0.4, 0.1, 0.6), (0.3, 0.5, 0.1), (0.6, 0.4, 0.2) \\ (0.2, 0.3, 0.6), (0.3, 0.7, 0.4), (0.4, 0.2, 0.4), (0.5, 0.3, 0.7) \\ (0.6, 0.3, 0.2), (0.5, 0.4, 0.1), (0.5, 0.4, 0.5), (0.4, 0.2, 0.6) \end{bmatrix}$$

In the following, we put on the scheme of assessment of five possible alternatives.

Step 1: We did practice the data assumed in the matrix $\tilde{E}$ and put on SFWMSM to discover the ideals of $\tilde{e}_i$ of the alternatives (Assume that $t = 2$):

$$\tilde{e}_1 = (0.7659, 0.11502, 0.2307),$$

$$\tilde{e}_2 = (0.8307, 0.2373, 0.1753),$$

$$\tilde{e}_3 = (0.8196, 0.1758, 0.1360),$$

$$\tilde{e}_4 = (0.7847, 0.1761, 0.2915),$$

$$\tilde{e}_5 = (0.8452, 0.1562, 0.2115)$$

Step 2: Find scores $S(\tilde{e}_i)(i = 1, 2, \cdots, 5)$ of all spherical fuzzy values $\tilde{e}_i(i = 1, 2, \cdots, 5)$.

Step 3: Order all the values according to the scores.
In Table 1, we obtain the scores of the alternatives and find the best out of them. As we take $T = 1$, we find the alternate is best, and for $T = 2$, is also best, and for $T = 3$, is also best, and for $T = 4$, is the best alternate. From here we can see that with different values of T, we found different best alternatives. To replicate the effect of altered values of constraint T, we practice them in our suggested technique for ordering the alternatives. The outcome is graphically represented in Figure 1. Graph of the obtained ranking at the different values of parameter T are shown in Figure 1.

By Figure 1, it can be seen that the constraint T varies according to the choice creator's liking, and the standing outcomes are a little bit altered, which displays that the SFWMSM operator can reveal the choice creator's liking.

6.1. Comparative analysis

Now we relate our suggested technique with spherical fuzzy weighted aggregation (SFWA) (Ahmmad et al., 2021) and spherical fuzzy weighted geometric (SFWG), operators as below:

Definition 12. Let $\tilde{\alpha}_q = (\omega_q, \eta_q, \psi_q)(q = 1, 2, \ldots, n)$ be a collection of SFVs, $w = (w_1, w_2, \cdots, w_n)$ is taken as the weight vector of $\tilde{\alpha}_q(q = 1, 2, \cdots, n)$, and $w_q > 0$, $\sum_{q=1}^{n} w_q = 1$. Then,

$$SFWA_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\sqrt{1 - \prod_{q=1}^{n}(1 - (\omega_q)^2)^{w_q}}, \prod_{q=1}^{n}(\eta_q)^{w_q}, \prod_{q=1}^{n}(\psi_q)^{w_q}\right)$$

$$SFWG_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \begin{pmatrix} \prod_{q=1}^{n}(\omega_q)^{w_q}, \sqrt{1 - \prod_{q=1}^{n}(1 - (\eta_q)^2)^{w_\eta}}, \\ \sqrt{1 - \prod_{q=1}^{n}(1 - (\psi_q)^2)^{w_{jq}}} \end{pmatrix}$$

By utilizing the data of matrix $\tilde{E}$, we find the scores of alternatives in Table 2 and give the orders in Table 3. Outcomes are graphically represented in Figure 2. From Figure 2, the comparison between different AOs can be noted. The ranking of the alternatives obtained

Table 1
Score value of the SFWMSM and ranking of the alternate

						Ranking
$T = 1$	0.6186	0.7074	0.6909	0.6319	0.7373	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
$T = 2$	0.5854	0.6882	0.6715	0.6104	0.7123	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
$T = 3$	0.5703	0.6770	0.6626	0.6041	0.6953	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_1$
$T = 4$	0.9653	0.9390	0.9477	0.9401	0.9284	$\mathcal{A}_1 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_2 \succ \mathcal{A}_5$

Figure 1
Graphical representation of SFWMSM operator at different values of parameter

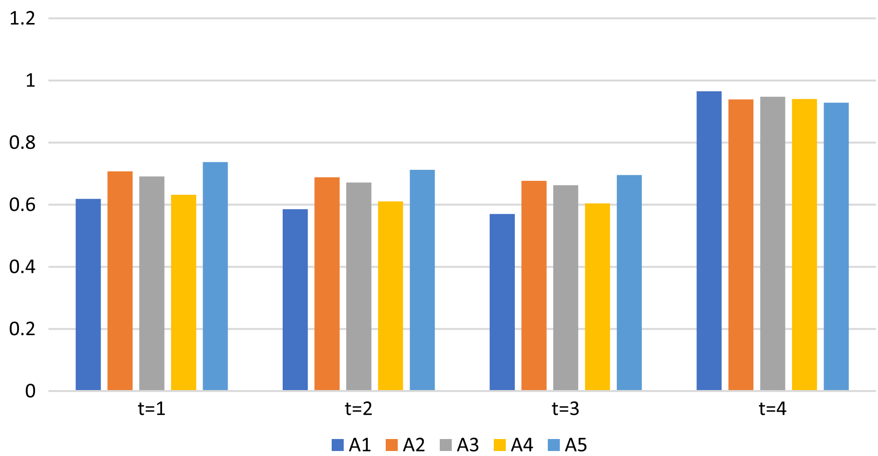


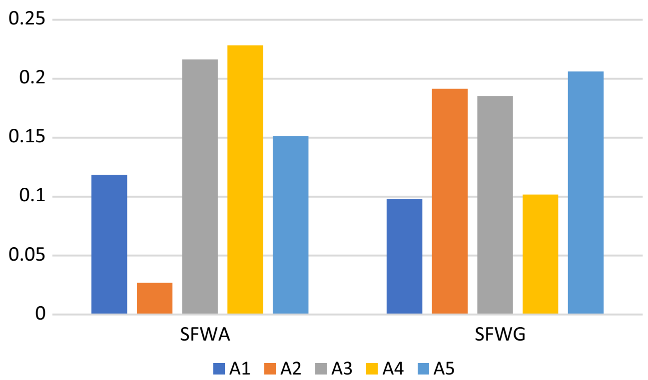
Table 2
Score function of the alternatives obtained by different AOs

SFWA	SFWG
0.1186	0.0982
0.0269	0.1915
0.2163	0.1854
0.2283	0.1018
0.1515	0.2061

Table 3
Ranking the above values obtained by different AOs

	Ordering
SFWA	$\mathcal{A}4 \succ \mathcal{A}3 \succ \mathcal{A}5 \succ \mathcal{A}1 \succ \mathcal{A}2$
SFWG	$\mathcal{A}_5 \succ \mathcal{A}_2 \succ \mathcal{A}_3 \succ \mathcal{A}_4 \succ \mathcal{A}_1$

Figure 2
Figure of score values obtained from SFWA and SFWG operator



from SFWA and SFWG operators is shown graphically. It can be noted that is the best alternative obtained from SFWG operator and is the best alternative obtained from SFWA operator.

7. Conclusions

In this study, we extended the MSM operator to the framework of the SFS. Consequently, a family of the AOs which includes the SFMSM and SFWMSM operators is developed. Some basic properties of the proposed AOs are investigated. The proposed AOs are applied to the MADM problem using spherical fuzzy information in this research. Then,

1. The obtained results have been observed at different values of the parameter involved and found that the variation of the parameter does not affect the ranking.
2. The obtained results from the proposed AOs are compared with some existing AOs and found that the obtained results are significant and the most generalized form if the existing AOs.
3. The comparison and the variation of the ranking results are tabulated and represented graphically. We aim to extend the MSM to frameworks defined in Akram and Khan (2021), Akram and Naz (2019), and Al-Quran (2021).

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Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

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