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Linear Diophantine Fuzzy Aggregation Operators with Multi-Criteria Decision-Making

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Abstract: A linear Diophantine fuzzy set (LDFS) is a new mathematical tool that can be used for optimization, artificial intelligence, and process modeling. The LDFS theory widens the area of fuzzy information via "reference parameters" due to its wonderful characteristic of a broad depiction zone for allowed doublets. Because the actual world is not exact, and there is a dearth of knowledge, determining and selecting the optimal choice is a tough and unforeseen decision-making dilemma. The primary goal is to guide decision-makers through the process of selecting the best option inside a linear Diophantine fuzzy context. We suggested two new aggregation operators: the "linear Diophantine fuzzy weighted average operator and the linear Diophantine fuzzy weighted geometric operator." Following that, the proposed model is validated using a clear example of linear Diophantine fuzzy content. This demonstrates the utility and applicability of the suggested strategy.

Keywords: aggregation operators, linear Diophantine fuzzy numbers and MCDM

1. Introduction

Since the early twentieth century, one of society's most urgent issues has been ambiguous and deceptive information. Data aggregation is critical for decision-making in a wide range of fields, including economics, management, sociology, science, technology, cognitive systems, and autonomous systems. People have traditionally understood knowledge of the alternative to be a definite quantity or linguistic number. However, due to the degree of ambiguity involved, the information is difficult to synthesist. The "multi-criteria decision-making" (MCDM) approach is a frequently used intellectual activity instrument whose primary objective is to pick from a restricted number of possibilities based on the details provided by decision-makers (DMs). On the other hand, the MCDM technique frequently results in unclear and erroneous results due to its propensity for both. This is due to the fact that it incorporates the complexities of cognitive reasoning ability, which makes it challenging for DMs to participate in the evaluation process in a correct fashion. In addition to addressing the issue of uncertainty, Zadeh (1965) was a pioneer in developing fuzzy set (FS) theory. It is imperative that a solution be found for this issue. Atanassov (1986) gave the notion of "intuitionistic fuzzy set (IFS)." Yager (2014) proposed "Pythagorean fuzzy set (PFS)" as an extended form of IFS. Yager (2017) added some generalizations to the IFS and PFS, and he developed the concept of the "q-rung orthopair fuzzy set (q-ROFS)." A constraint of the q-ROFS is that the sum of qth "membership degree" (MSD) power and "non-membership degree" (NMSD) power must be equal to or less than one. Riaz and Hashmi (2019) established the notion of the LDFS . After the advent of this notion, a number of academics were drawn to it and began working in this field.

Xu (2007) and Xu and Xia (2011) gave some AOs related to IFS. Wei et al. (2014), Mahmood et al. (2017), Feng et al. (2019), Zhang et al. (2014), and Zhao et al. (2010) offered several AOs for various FS extensions. Alcantud et al. (2022) proposed some AOs for N-soft sets. Feng et al. (2022) proposed some novel score functions related to orthopair fuzzy set. Senapati and Yager (2020) proposed Fermatean fuzzy set as the extension of IFS. Smarandache (1999) and Wang et al. (2010) proposed a novel idea of neutrosophic set. Ashraf et al. (2019; 2020) proposed some distance metric for "cubic picture fuzzy set." Saha, et al. (2021a), and Saha, et al. (2021b) introduced some hybrid AOs for different extensions of fuzzy set. Wei and Zhang (2019) gave some Bonferroni power AOs. Riaz et al. (2021a) proposed a number of AOs, including interactive and prioritized with PDs Riaz et al. (2021b). Some extra-ordinary work related to proposed work is given in Karaaslan and Ozlu (2020); Din et al. (2022); Gul et al. (2022); Alcantud (2022). Akram et al. (2021) gave the idea of Pythagorean ELECTRE-II approach. Garg et al. (2022) proposed VIKOR approach. Garg and Kaur (2022) gave the notion of correlation coefficients under cubic intuitionistic fuzzy set. Khan et al. (2022) introduced some complex T-spherical fuzzy AOs. Pramanik and Dalapati (2022) proposed VIKOR approach for bipolar neutrosophic set. Liu and Wang (2018) proposed some basic AOs for q-ROFSs.

This format is maintained for the remainder of the paper. In the second portion, we will talk about some essential LDFS concepts. The third section offers several potential AOs for LDFNs.

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In Section 4, an MCDM framework is shown for the recommended AOs. Section 5 has a test scenario with numerical information. The most important findings from the research are discussed in the sixth section.

2. Basic Definitions

In this part, we will go over some of the most fundamental aspects of LDFS.

Definition 2.1. Riaz and Hashmi (2019) A LDFS R^r in X can be characterized by

$$R^r = \{(\text{Ш}, \langle \zeta^\tau_{R^r}(\text{Ш}), \daleth^\upsilon_{R^r}(\text{Ш})\rangle, \langle \xi^\hbar_{R^r}(\text{Ш}), \beth^\gamma_{R^r}(\text{Ш})\rangle) : \text{Ш} \in X\},$$

where $\zeta^\tau{}_{R^r}(\text{Ш}), \daleth^\upsilon{}_{R^r}(\text{Ш}), \xi^\hbar{}_{R^r}(\text{Ш}), \beth^\gamma{}_{R^r}(\text{Ш}) \in [0,1]$ are the MSD, the NMSD and the corresponding RPs. Furthermore,

$$0 \leq \xi^\hbar_{R^r}(\text{Ш}) + \beth^\gamma_{R^r}(\text{Ш}) \leq 1,$$

and

$$0 \leq \xi^\hbar_{R^r}(\text{Ш})\zeta^\tau_{R^r}(\text{Ш}) + \beth^\gamma_{R^r}(\text{Ш})\daleth^\upsilon_{R^r}(\text{Ш}) \leq 1$$

for all $\text{Ш} \in X$. The LDFS

$$R^r_X = \{(\text{Ш}, \langle 1, 0\rangle, \langle 1, 0\rangle) : \text{Ш} \in \text{X}\}$$

is recognized the "absolute LDFS" in X. The LDFS

$$R^r_\phi = \{(\text{Ш}, \langle 0, 1\rangle, \langle 0, 1\rangle) : \text{Ш} \in \text{X}\}$$

is recognized the "null LDFS" in X.

Modeling or categorization certain structures can be accomplished with the help of the RPs. We are able to describe a wide variety of systems by altering the fundamental significance of the RPs. Moreover, $\daleth_{R^r}(\text{Ш})\ \pi_{R^r}(\text{Ш}) = 1 - (\xi^\hbar{}_{R^r}(\text{Ш})\ \zeta^\tau{}_{R^r}(\text{Ш}) + \beth^\gamma{}_{R^r}(\text{Ш})\daleth^\upsilon{}_{R^r}(\text{Ш}))$ is called the "indeterminacy degree" and its corresponding RP of Ш to R^r.

It is very evident that our suggested conception is more appropriate and advanced than that of someone else, and it includes a range of RPs. This procedure is applicable to a wide range of projects, including those in the fields of industry, medicine, cognitive computing, and MCDM.

Definition 2.2. Riaz and Hashmi (2019) A "linear Diophantine fuzzy number" (LDFN) is the form of $\varsigma^\vartheta = (\langle \zeta^\tau{}_{\varsigma^\vartheta}, \daleth^\upsilon{}_{\varsigma^\vartheta}\rangle, \langle \xi^\hbar{}_{\varsigma^\vartheta}, \beth^\gamma{}_{\varsigma^\vartheta}\rangle)$ having the given characteristics:

(1) $0 \leq \zeta^\tau{}_{\varsigma^\vartheta}, \daleth^\upsilon{}_{\varsigma^\vartheta}, \xi^\hbar{}_{\varsigma^\vartheta}, \beth^\gamma{}_{\varsigma^\vartheta} \leq 1$;
(2) $0 \leq \xi^\hbar{}_{\varsigma^\vartheta} + \beth^\gamma{}_{\varsigma^\vartheta} \leq 1$;
(3) $0 \leq \xi^\hbar{}_{\varsigma^\vartheta}\zeta^\tau{}_{\varsigma^\vartheta} + \beth^\gamma{}_{\varsigma^\vartheta}\daleth^\upsilon{}_{\varsigma^\vartheta} \leq 1$.

Definition 2.3. Riaz and Hashmi (2019) Consider $\varsigma^\vartheta = (\langle \zeta^\tau{}_{\varsigma^\vartheta}, \daleth^\upsilon{}_{\varsigma^\vartheta}\rangle, \langle \xi^\hbar{}_{\varsigma^\vartheta}, \beth^\gamma{}_{\varsigma^\vartheta}\rangle)$ is the LDFN, then the "score function" (SF) $\maltese(\varsigma^\vartheta)$ is defined by $\maltese(\varsigma^\vartheta) : LDFN(X) \to [-1,1]$ and given by

$$\maltese(\varsigma^\vartheta) = \frac{1}{2}[(\zeta^\tau{}_{\varsigma^\vartheta} - \daleth^\upsilon{}_{\varsigma^\vartheta}) + (\xi^\hbar{}_{\varsigma^\vartheta} - \beth^\gamma{}_{\varsigma^\vartheta})]$$

where $\mathrm{LDFN}(X)$ is the collection of LDFNs on X.

Definition 2.4. Riaz and Hashmi (2019) Consider $\varsigma^\vartheta = (\langle \zeta^\tau{}_{\varsigma^\vartheta}, \daleth^\upsilon{}_{\varsigma^\vartheta}\rangle, \langle \xi^\hbar{}_{\varsigma^\vartheta}, \beth^\gamma{}_{\varsigma^\vartheta}\rangle)$ is the LDFN, then the "accuracy function" is defined by $\psi : LDFN(X) \to [0,1]$ and given as

$$\psi(\varsigma^\vartheta) = \frac{1}{2}\left[\left(\frac{\zeta^\tau{}_{\varsigma^\vartheta} + \daleth^\upsilon{}_{\varsigma^\vartheta}}{2}\right) + (\xi^\hbar{}_{\varsigma^\vartheta} + \beth^\gamma{}_{\varsigma^\vartheta})\right]$$

Definition 2.5. Riaz and Hashmi (2019) Consider $\varsigma^\vartheta{}_1 = (\langle \zeta^\tau{}_1, \daleth^\upsilon{}_1\rangle, \langle \xi^\hbar{}_1, \beth^\gamma{}_1\rangle)$ is the LDFN and $\text{Ӿ} > 0$. Then,

- $\varsigma^{\vartheta^c}{}_1 = (\langle \daleth^\upsilon{}_1, \zeta^\tau{}_1\rangle, \langle \beth^\gamma{}_1, \xi^\hbar{}_1\rangle)$;
- $\text{Ӿ}\varsigma^\vartheta{}_1 = (\langle 1-(1-\zeta^\tau{}_1)^{\text{Ӿ}}, \daleth^{\upsilon^{\text{Ӿ}}}_1\rangle, \langle 1-(1-\xi^\hbar{}_1)^{\text{Ӿ}}, \beth^{\gamma^{\text{Ӿ}}}_1\rangle)$;
- $\varsigma^{\vartheta^{\text{Ӿ}}}_1 = (\langle \zeta^{\tau^{\text{Ӿ}}}_1, 1-(1-\daleth^\upsilon{}_1)^{\text{Ӿ}}\rangle, \langle \xi^{\hbar^{\text{Ӿ}}}_1, 1-(1-\beth^\gamma{}_1)^{\text{Ӿ}}\rangle)$.

Definition 2.6. Riaz and Hashmi (2019) Consider $\varsigma^\vartheta{}_i = (\langle \zeta^\tau{}_i, \daleth^\upsilon{}_i\rangle, \langle \xi^\hbar{}_i, \beth^\gamma{}_i\rangle)$ is the LDFNs with $i = 1, 2$. Then,

- $\varsigma^\vartheta{}_1 \subseteq \varsigma^\vartheta{}_2 \Leftrightarrow \zeta^\tau{}_1 \leq \zeta^\tau{}_2, \daleth^\upsilon{}_2 \leq \daleth^\upsilon{}_1, \xi^\hbar{}_1 \leq \xi^\hbar{}_2, \beth^\gamma{}_2 \leq \beth^\gamma{}_1$;
- $\varsigma^\vartheta{}_1 = \varsigma^\vartheta{}_2 \Leftrightarrow \zeta^\tau{}_1 = \zeta^\tau{}_2, \daleth^\upsilon{}_1 = \daleth^\upsilon{}_2, \xi^\hbar{}_1 = \xi^\hbar{}_2, \beth^\gamma{}_1 = \beth^\gamma{}_2$;
- $\varsigma^\vartheta{}_1 \oplus \varsigma^\vartheta{}_2 = (\langle \zeta^\tau{}_1 + \zeta^\tau{}_2 - \zeta^\tau{}_1\zeta^\tau{}_2, \daleth^\upsilon{}_1\daleth^\upsilon{}_2\rangle, \langle \xi^\hbar{}_1 + \xi^\hbar{}_2 - \xi^\hbar{}_1\xi^\hbar{}_2, \beth^\gamma{}_1\beth^\gamma{}_2\rangle)$;
- $\varsigma^\vartheta{}_1 \otimes \varsigma^\vartheta{}_2 = (\langle \zeta^\tau{}_1\zeta^\tau{}_2, \daleth^\upsilon{}_1 + \daleth^\upsilon{}_2 - \daleth^\upsilon{}_1\daleth^\upsilon{}_2\rangle, \langle \xi^\hbar{}_1\xi^\hbar{}_2, \beth^\gamma{}_1 + \beth^\gamma{}_2 - \beth^\gamma{}_1\beth^\gamma{}_2\rangle)$.

Definition 2.7. Riaz and Hashmi (2019) Consider $\varsigma^\vartheta{}_i = (\langle \zeta^\tau{}_i, \daleth^\upsilon{}_i\rangle, \langle \xi^\hbar{}_i, \beth^\gamma{}_i\rangle)$ is the assemblage of LDFNs with $i \in \Delta$. Then,

- $\bigcup_{i\in\Delta} \varsigma^\vartheta{}_i = (\langle \sup_{i\in\Delta} \zeta^\tau{}_i, \inf_{i\in\Delta} \daleth^\upsilon{}_i\rangle, \langle \sup_{i\in\Delta} \xi^\hbar{}_i, \inf_{i\in\Delta} \beth^\gamma{}_i\rangle)$;
- $\bigcap_{i\in\Delta} \varsigma^\vartheta{}_i = (\langle \inf_{i\in\Delta} \zeta^\tau{}_i, \sup_{i\in\Delta} \daleth^\upsilon{}_i\rangle, \langle \inf_{i\in\Delta} \xi^\hbar{}_i, \sup_{i\in\Delta} \beth^\gamma{}_i\rangle)$.

We offer a graphical depiction of LDFS with a number of various RP configurations and explain how its assessment space is bigger than that of IFS and PFS by showing how its assessment space is shown with a variety of different RP combinations. Figures 1, 2 and 3 show the comparison of IFS, PFS and LDFS, while Figures 4, 5 and 6 show the graphical view of LDFS with different pairs of constant RPs.

Figure 1
Graphical depiction of IFSs

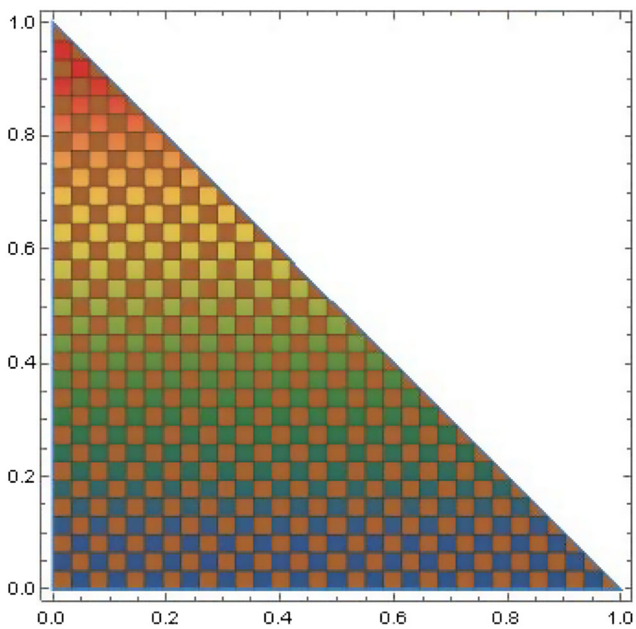


Figure 2
Graphical depiction of PFSs

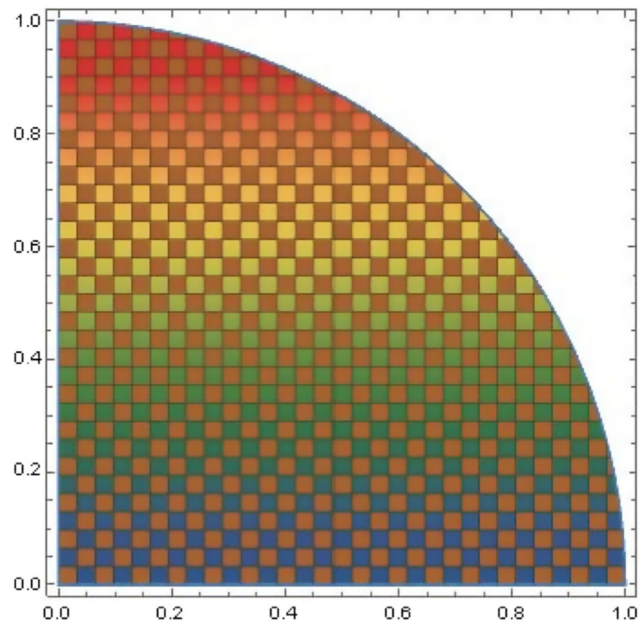
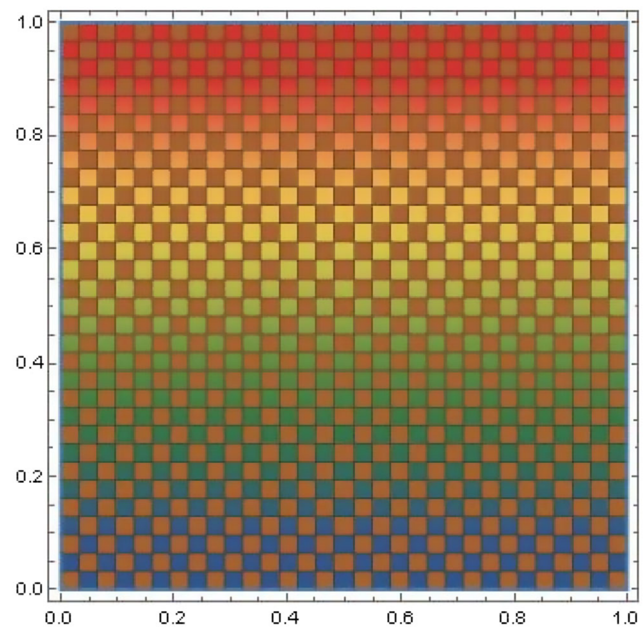


Figure 3
Graphical depiction of LDFSs



3. Linear Diophantine Fuzzy Aggregation Operators

In this section, we discussed LDFWA operator and LDFWG operator.

3.1. LDFWA operator

Definition 3.1. Consider $\varsigma^{\vartheta}_{\gimel} = (\langle \zeta^{\tau}_{\gimel}, \daleth^{\upsilon}_{\gimel}\rangle, \langle \xi^{\hbar}_{\gimel}, \beth^{\gamma}_{\gimel}\rangle)$ is the collection of LDFNs, and LDFWA : $\Bbbk^{n} \to \Bbbk$, be the mapping.

$$\mathrm{LDFWA}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \ldots \varsigma^{\vartheta}{}_n) = \mathfrak{P}^{\gamma}{}_1 \varsigma^{\vartheta}{}_1 \oplus \mathfrak{P}^{\gamma}{}_2 \varsigma^{\vartheta}{}_2 \oplus \ldots, \oplus \mathfrak{P}^{\gamma}{}_n \varsigma^{\vartheta}{}_n \quad (1)$$

Figure 4
LDFS with $\langle \xi^{h}, \beth^{\gamma}\rangle = \langle 0.3, 0.4\rangle$

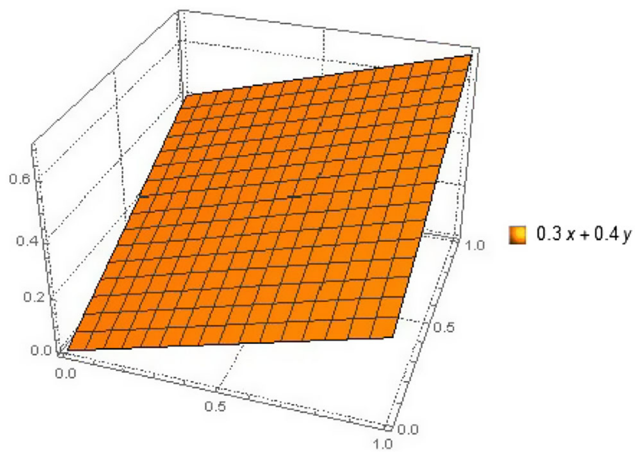


Figure 5
LDFS with $\langle \xi^{h}, \beth^{\gamma}\rangle = \langle 0.8, 0.1\rangle$

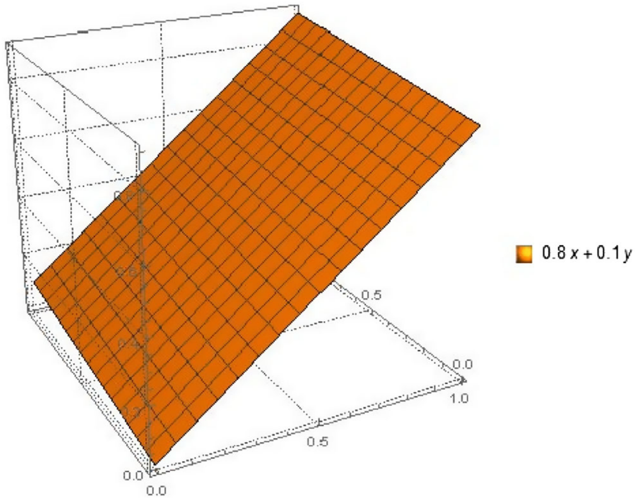
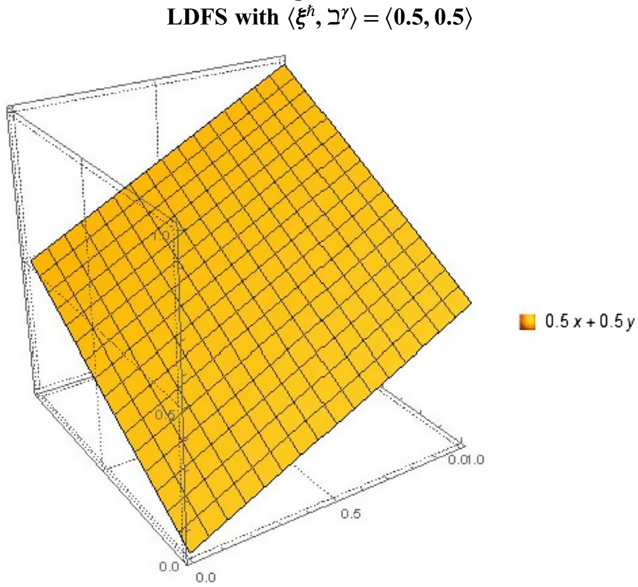


Figure 6
LDFS with $\langle \xi^{h}, \beth^{\gamma}\rangle = \langle 0.5, 0.5\rangle$



Then, LDFWA is known as LDFWA operator, where $(\mathfrak{P}^{\gamma}_1, \mathfrak{P}^{\gamma}_2, \ldots, \mathfrak{P}^{\gamma}_n)$ be the weight vector (WV) with the constraint $\mathfrak{P}^{\gamma}_h > 0$ and $\sum_{h=1}^{n} \mathfrak{P}^{\gamma}_h = 1$.

We might also think about LDFWA operator by employing the theorem following.

Theorem 3.1. Consider $\varsigma^{\vartheta}_{\gimel} = (\langle \zeta^{\tau}_{\gimel}, \daleth^{\upsilon}_{\gimel} \rangle, \langle \xi^{\hbar}_{\gimel}, \beth^{\gamma}_{\gimel} \rangle)$ is the collection of LDFNs, we also evaluate LDFWA by

$$\mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_n) = \left(\left\langle 1 - \prod_{\gimel=1}^{n} (1 - \zeta^{\tau}_{j})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{n} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{j}}_{\gimel} \right\rangle, \left\langle 1 - \prod_{\gimel=1}^{n} (1 - \xi^{\hbar}_{\gimel})^{\mathfrak{P}^{\gamma}_{j}}, \prod_{\gimel=1}^{n} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{j}}_{\gimel} \right\rangle \right) \tag{2}$$

Proof. It is quite simple for the first assertion to come before the Definition 3.1 and the Theorem 3.1. The following instances demonstrate this point further:

$$\mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_n) = \left(\mathfrak{P}^{\gamma}_1 \varsigma^{\vartheta}_1 \oplus \mathfrak{P}^{\gamma}_2 \varsigma^{\vartheta}_2 \oplus \ldots, \mathfrak{P}^{\gamma}_n \varsigma^{\vartheta}_n \right)$$
$$= \left(\left\langle 1 - \prod_{\gimel=1}^{n} (1 - \zeta^{\tau}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{n} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle, \left\langle \prod_{\gimel=1}^{n} (1 - \xi^{\hbar}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{n} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle \right)$$

In order to demonstrate the validity of this theorem, we turned to mathematics induction.
For $n = 2$

$$\mathfrak{P}^{\gamma}_1 \varsigma^{\vartheta}_1 = \left(\left\langle 1 - (1 - \zeta^{\tau}_1)^{\mathfrak{P}^{\gamma}_1}, \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_1}_{1} \right\rangle, \left\langle 1 - (1 - \xi^{\hbar}_1)^{\mathfrak{P}^{\gamma}_1}, \beth^{\gamma\, \mathfrak{P}^{\gamma}_1}_{1} \right\rangle \right)$$

$$\mathfrak{P}^{\gamma}_2 \varsigma^{\vartheta}_2 = \left(\left\langle 1 - (1 - \zeta^{\tau}_2)^{\mathfrak{P}^{\gamma}_1}, \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_1}_{2} \right\rangle, \left\langle 1 - (1 - \xi^{\hbar}_2)^{\mathfrak{P}^{\gamma}_1}, \beth^{\gamma\, \mathfrak{P}^{\gamma}_1}_{2} \right\rangle \right)$$

Then,

$$\mathfrak{P}^{\gamma}_1 \varsigma^{\vartheta}_1 \oplus \mathfrak{P}^{\gamma}_2 \varsigma^{\vartheta}_2$$
$$= \left(\left\langle 1 - (1 - \zeta^{\tau}_1)^{\mathfrak{P}^{\gamma}_1}, \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_1}_{1} \right\rangle, \left\langle 1 - (1 - \xi^{\hbar}_1)^{\mathfrak{P}^{\gamma}_1}, \beth^{\gamma\, \mathfrak{P}^{\gamma}_1}_{1} \right\rangle \right) \oplus$$
$$\left(\left\langle 1 - (1 - \zeta^{\tau}_2)^{\mathfrak{P}^{\gamma}_1}, \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_1}_{2} \right\rangle, \left\langle 1 - (1 - \xi^{\hbar}_2)^{\mathfrak{P}^{\gamma}_1}, \beth^{\gamma\, \mathfrak{P}^{\gamma}_1}_{2} \right\rangle \right)$$
$$= \left(\left\langle 1 - (1 - \zeta^{\tau}_1)^{\mathfrak{P}^{\gamma}_1} + 1 - (1 - \zeta^{\tau}_2)^{\mathfrak{P}^{\gamma}_1} - \left((1 - (1 - \zeta^{\tau}_1)^{\mathfrak{P}^{\gamma}_1} \right) \left((1 - (1 - \zeta^{\tau}_2)^{\mathfrak{P}^{\gamma}_1} \right), \right.\right.$$
$$\left. \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_1}_{1} . \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_1}_{2} \right\rangle, \left\langle 1 - (1 - \xi^{\hbar}_1)^{\mathfrak{P}^{\gamma}_1} + 1 - (1 - \xi^{\hbar}_2)^{\mathfrak{P}^{\gamma}_1} - \left((1 - (1 - \xi^{\hbar}_1)^{\mathfrak{P}^{\gamma}_1} \right) \right.$$
$$\left.\left. \left(1 - (1 - \xi^{\hbar}_2)^{\mathfrak{P}^{\gamma}_1} \right), \quad \beth^{\gamma\, \mathfrak{P}^{\gamma}_1}_{1} . \beth^{\gamma\, \mathfrak{P}^{\gamma}_1}_{2} \right\rangle \right)$$
$$= \left(\left\langle 1 - (1 - \zeta^{\tau}_1)^{\mathfrak{P}^{\gamma}_1} (1 - \zeta^{\tau}_2)^{\mathfrak{P}^{\gamma}_1}, \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_1}_{1} . \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_1}_{2} \right\rangle, \right.$$
$$\left. \left\langle 1 - (1 - \xi^{\hbar}_1)^{\mathfrak{P}^{\gamma}_1} (1 - \xi^{\hbar}_2)^{\mathfrak{P}^{\gamma}_1}, \beth^{\gamma\, \mathfrak{P}^{\gamma}_1}_{1} . \beth^{\gamma\, \mathfrak{P}^{\gamma}_1}_{2} \right\rangle \right)$$
$$= \left(\left\langle 1 - \prod_{\gimel=1}^{2} (1 - \zeta^{\tau}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{2} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle, \left\langle 1 - \prod_{\gimel=1}^{2} (1 - \xi^{\hbar}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{2} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle \right)$$

This demonstrates that the Equation 2 is correct for the value of n equal to two; now assuming that the Equation 2 is accurate for the value of n equal to k, that is,

$$\mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_k) = \left(\left\langle 1 - \prod_{\gimel=1}^{k} (1 - \zeta^{\tau}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{k} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle, \left\langle 1 - \prod_{\gimel=1}^{k} (1 - \xi^{\hbar}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{k} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle \right)$$

Now that "$n = k + 1$", according to the operational laws that govern LDFNs, we obtain,

$$\mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_{k+1}) = \mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_k) \oplus \mathfrak{P}^{\gamma}_{\gimel} \varsigma^{\vartheta}_{k+1}$$
$$= \left(\left\langle 1 - \prod_{\gimel=1}^{k} (1 - \zeta^{\tau}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{k} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle, \left\langle 1 - \prod_{\gimel=1}^{k} (1 - \xi^{\hbar}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{k} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle \right) \oplus$$
$$\left(\left\langle 1 - (1 - \zeta^{\tau}_{k+1})^{\mathfrak{P}^{\gamma}_{k+1}}, \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{k+1}}_{k+1} \right\rangle, \left\langle 1 - (1 - \xi^{\hbar}_{k+1})^{\mathfrak{P}^{\gamma}_{k+1}}, \beth^{\gamma\, \mathfrak{P}^{\gamma}_{k+1}}_{k+1} \right\rangle \right)$$
$$= \left(\left\langle 1 - \prod_{\gimel=1}^{k} (1 - \zeta^{\tau}_{k})^{\mathfrak{P}^{\gamma}_{\gimel}} + 1 - (1 - \zeta^{\tau}_{k+1})^{\mathfrak{P}^{\gamma}_{k+1}} - \left(1 - \prod_{\gimel=1}^{k} (1 - \zeta^{\tau}_{k})^{\mathfrak{P}^{\gamma}_{\gimel}} \right) \left(1 - (1 - \zeta^{\tau}_{k+1})^{\mathfrak{P}^{\gamma}_{k+1}} \right), \right.\right.$$
$$\left. \prod_{\gimel=1}^{k} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{\gimel}}_{k} . \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{k+1}}_{k+1} \right\rangle, \left\langle 1 - \prod_{\gimel=1}^{k} (1 - \xi^{\hbar}_{k})^{\mathfrak{P}^{\gamma}_{\gimel}} + 1 - (1 - \xi^{\hbar}_{k+1})^{\mathfrak{P}^{\gamma}_{k+1}} - \right.$$
$$\left.\left. \left(1 - \prod_{\gimel=1}^{k} (1 - \xi^{\hbar}_{k})^{\mathfrak{P}^{\gamma}_{\gimel}} \right) \left(1 - (1 - \xi^{\hbar}_{k+1})^{\mathfrak{P}^{\gamma}_{k+1}} \right), \prod_{\gimel=1}^{k} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{\gimel}}_{k} \,.\, \beth^{\gamma\, \mathfrak{P}^{\gamma}_{k+1}}_{k+1} \right\rangle \right)$$
$$= \left(\left\langle 1 - \prod_{\gimel=1}^{k} (1 - \zeta^{\tau}_{k})^{\mathfrak{P}^{\gamma}_{\gimel}} (1 - \zeta^{\tau}_{k+1})^{k+1}, \prod_{\gimel=1}^{k} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{\gimel}}_{k} \,.\, \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{k+1}}_{k+1} \right\rangle, \right.$$
$$\left. \left\langle 1 - \prod_{\gimel=1}^{k} (1 - \xi^{\hbar}_{k})^{\mathfrak{P}^{\gamma}_{\gimel}} (1 - \xi^{\hbar}_{k+1})^{k+1}, \prod_{\gimel=1}^{k} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{\gimel}}_{k} \,.\, \beth^{\gamma\, \mathfrak{P}^{\gamma}_{k+1}}_{k+1} \right\rangle \right)$$
$$= \left(\left\langle 1 - \prod_{\gimel=1}^{k+1} (1 - \zeta^{\tau}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{k+1} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle, \left\langle 1 - \prod_{\gimel=1}^{k+1} (1 - \xi^{\hbar}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{k+1} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle \right)$$

This demonstrates that the Equation 2 is valid for the value of $n = k + 1$. Then,

$$\mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_n) = \left(\left\langle 1 - \prod_{\gimel=1}^{n} (1 - \zeta^{\tau}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{n} \daleth^{\upsilon\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle, \left\langle 1 - \prod_{\gimel=1}^{n} (1 - \xi^{\hbar}_{\gimel})^{\mathfrak{P}^{\gamma}_{\gimel}}, \prod_{\gimel=1}^{n} \beth^{\gamma\, \mathfrak{P}^{\gamma}_{\gimel}}_{\gimel} \right\rangle \right)$$

□

The next couple of paragraphs will discuss a few of the beneficial qualities that LDFWA operator has.

Theorem 3.2. Consider $\varsigma^{\vartheta}_{\gimel} = (\langle \zeta^{\tau}_{\gimel}, \daleth^{\upsilon}_{\gimel} \rangle, \langle \xi^{\hbar}_{\gimel}, \beth^{\gamma}_{\gimel} \rangle)$ is the collection of LDFNs. If all $\varsigma^{\vartheta}_{\gimel}$ are equal, that is, $\varsigma^{\vartheta}_{\gimel} = \varsigma^{\vartheta}\ \forall j$, then

$$\mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_{\mathrm{n}}) = \varsigma^{\vartheta}$$

Proof. From Definition 3.1, we have

$$\begin{aligned} \mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_n) &= \mathfrak{P}^{\gamma}_1 \varsigma^{\vartheta}_1 \oplus \mathfrak{P}^{\gamma}_2 \varsigma^{\vartheta}_2 \oplus \ldots, \oplus \mathfrak{P}^{\gamma}_n \varsigma^{\vartheta}_n \\ &= \mathfrak{P}^{\gamma}_1 \varsigma^{\vartheta} \oplus \mathfrak{P}^{\gamma}_2 \varsigma^{\vartheta} \oplus \ldots, \oplus \mathfrak{P}^{\gamma}_n \varsigma^{\vartheta} \\ &= (\mathfrak{P}^{\gamma}_1 + \mathfrak{P}^{\gamma}_2 + \ldots + \mathfrak{P}^{\gamma}_n) \varsigma^{\vartheta} \\ &= \varsigma^{\vartheta} \end{aligned}$$

□

Corollary 3.1. If $\varsigma^{\vartheta}_{\gimel} = (\langle \zeta^{\tau}_{\gimel}, \daleth^{\upsilon}_{\gimel} \rangle, \langle \xi^{\hbar}_{\gimel}, \beth^{\gamma}_{\gimel} \rangle)$ is the collection of absolute LDFNs, that is, $\varsigma^{\vartheta}_{\gimel} = \langle (1,0), (1,0) \rangle$ for all j, then

$$\mathrm{LDFWA}(\varsigma^{\vartheta}_1, \varsigma^{\vartheta}_2, \ldots \varsigma^{\vartheta}_{\mathrm{n}}) = \langle (1,0), (1,0) \rangle$$

Proof. It should not be difficult for us to find a corollary that is analogous to the Theorem 3.2. □

Theorem 3.3. (Monotonicity) Assume that $\varsigma^{\vartheta}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \daleth^{\upsilon}{}_{\gimel} \rangle, \langle \xi^{\hbar}{}_{\gimel}, \beth^{\gamma}{}_{\gimel} \rangle)$ and $\varsigma^{\vartheta *}{}_{\gimel} = (\langle \zeta^{\tau *}{}_{\gimel}, \daleth^{\upsilon *}{}_{\gimel} \rangle, \langle \xi^{\hbar *}{}_{\gimel}, \beth^{\gamma *}{}_{\gimel} \rangle)$ are the assemblages of LDFNs, If $\zeta^{\tau *}{}_{\gimel} \geq \zeta^{\tau}{}_{\gimel}$, $\daleth^{\upsilon *}{}_{\gimel} \leq \daleth^{\upsilon}{}_{\gimel}$, $\xi^{\hbar *}{}_{\gimel} \geq \xi^{\hbar}{}_{\gimel}$ and $\beth^{\gamma *}{}_{\gimel} \leq \beth^{\gamma}{}_{\gimel}$ for all j, then

$$\text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{\mathrm{n}}) \leq \text{LDFWA}(\varsigma^{\vartheta *}_{1}, \varsigma^{\vartheta *}_{2}, \ldots \varsigma^{\vartheta *}_{\mathrm{n}})$$

Proof. Here, $\zeta^{\tau *}{}_{\gimel} \geq \zeta^{\tau}{}_{\gimel}$ and $\daleth^{\upsilon *}{}_{\gimel} \leq \daleth^{\upsilon}{}_{\gimel}$ for all j, If $\zeta^{\tau *}{}_{\gimel} \geq \zeta^{\tau}{}_{\gimel}$. $\Leftrightarrow \zeta^{\tau *}{}_{\gimel} \geq \zeta^{\tau}{}_{\gimel} \Leftrightarrow 1 - \zeta^{\tau *}{}_{\gimel} \leq 1 - \zeta^{\tau}{}_{\gimel}$

$$\Leftrightarrow (1 - \zeta^{\tau *}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (1 - \zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

$$\Leftrightarrow \overline{\prod}_{\gimel=1}^{n} (1 - \zeta^{\tau *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod}_{\gimel=1}^{n} (1 - \zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

$$\Leftrightarrow 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \zeta^{\tau *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

Again,
$\xi^{\hbar}{}_{\gimel}* \geq \xi^{\hbar}{}_{\gimel}$ and $\beth^{\gamma}{}_{\gimel}* \leq \beth^{\gamma}{}_{\gimel}$ for all j, If $\xi^{\hbar}{}_{\gimel}* \geq \xi^{\hbar}{}_{\gimel}$.
$\Leftrightarrow \xi^{\hbar}{}_{\gimel}* \geq \xi^{\hbar}{}_{\gimel} \Leftrightarrow 1 - \xi^{\hbar}{}_{\gimel}* \leq 1 - \xi^{\hbar}{}_{\gimel}$

$$\Leftrightarrow (1 - \xi^{\hbar}{}_{\gimel}*)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (1 - \xi^{\hbar}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

$$\Leftrightarrow \overline{\prod}_{\gimel=1}^{n} (1 - \xi^{\hbar *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod}_{\gimel=1}^{n} (1 - \xi^{\hbar}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

$$\Leftrightarrow 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \xi^{\hbar}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \xi^{\hbar *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

Now,
$\daleth^{\upsilon}{}_{\gimel}* \leq \daleth^{\upsilon}{}_{\gimel}$.
$\Leftrightarrow (\daleth^{\upsilon}{}_{\gimel}*)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (\daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$

$$\Leftrightarrow \overline{\prod}_{\gimel=1}^{n} (\daleth^{\upsilon *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod}_{\gimel=1}^{n} (\daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

And,
$\beth^{\gamma}{}_{\gimel}* \leq \beth^{\gamma}{}_{\gimel}$.
$\Leftrightarrow (\beth^{\gamma}{}_{\gimel}*)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (\beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$

$$\Leftrightarrow \overline{\prod}_{\gimel=1}^{n} (\beth^{\gamma *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod}_{\gimel=1}^{n} (\beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

Let

$$\overline{\varsigma^{\vartheta}} = \text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{\mathrm{n}})$$

and

$$\overline{\varsigma^{\vartheta *}} = \text{LDFWA}(\varsigma^{\vartheta *}_{1}, \varsigma^{\vartheta *}_{2}, \ldots \varsigma^{\vartheta *}_{\mathrm{n}})$$

We get that $\overline{\varsigma^{\vartheta *}} \geq \overline{\varsigma^{\vartheta}}$. So,

$$\text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{\mathrm{n}}) \leq \text{LDFWA}(\varsigma^{\vartheta *}_{1}, \varsigma^{\vartheta *}_{2}, \ldots \varsigma^{\vartheta *}_{\mathrm{n}})$$

Theorem 3.4. Assume that $\varsigma^{\vartheta}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \daleth^{\upsilon}{}_{\gimel} \rangle, \langle \xi^{\hbar}{}_{\gimel}, \beth^{\gamma}{}_{\gimel} \rangle)$ and $F^{\gamma}{}_{\gimel} = (\langle \phi_{\gimel}, \varphi_{\gimel} \rangle, \langle \mathscr{K}_{\gimel}, \mathscr{M}_{\gimel} \rangle)$ are two families of LDFNs. If $r > 0$ and $F^{\gamma} = (\langle \zeta^{\tau}{}_{F^{\gamma}}, \daleth^{\upsilon}{}_{F^{\gamma}} \rangle, \langle \xi^{\hbar}{}_{F^{\gamma}}, \beth^{\gamma}{}_{F^{\gamma}} \rangle)$ is a LDFN, then

1. $\text{LDFWA}(\varsigma^{\vartheta}{}_{1} \oplus F^{\gamma}, \varsigma^{\vartheta}{}_{2} \oplus F^{\gamma}, \ldots \varsigma^{\vartheta}{}_{n} \oplus F^{\gamma}) = \text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n}) \oplus F^{\gamma}$
2. $\text{LDFWA}(r\varsigma^{\vartheta}{}_{1}, r\varsigma^{\vartheta}{}_{2}, \ldots r\varsigma^{\vartheta}{}_{n}) = r\ \text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n})$
3. $\text{LDFWA}(\varsigma^{\vartheta}{}_{1} \oplus F^{\gamma}{}_{1}, \varsigma^{\vartheta}{}_{2} \oplus F^{\gamma}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n} \oplus F^{\gamma}{}_{n}) = \text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n}) \oplus \text{LDFWA}(F^{\gamma}{}_{1}, F^{\gamma}{}_{2}, \ldots F^{\gamma}{}_{n})$
4. $\text{LDFWA}(r\varsigma^{\vartheta}{}_{1} \oplus F^{\gamma}, r\varsigma^{\vartheta}{}_{2} \oplus F^{\gamma}, \ldots \oplus r\varsigma^{\vartheta}{}_{n} \oplus F^{\gamma}) = r\ \text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n}) \oplus F^{\gamma}$

Proof. Here, we just proof 1 and 3,
1.
Since

$$\varsigma^{\vartheta}{}_{\gimel} \oplus F^{\gamma} = \left(\left(1 - (1 - \zeta^{\tau}{}_{\gimel})(1 - \zeta^{\tau}{}_{F^{\gamma}}), \daleth^{\upsilon}{}_{\gimel} \daleth^{\upsilon}{}_{F^{\gamma}} \right), \left(1 - (1 - \xi^{\hbar}{}_{\gimel})(1 - \xi^{\hbar}{}_{F^{\gamma}}), \beth^{\gamma}{}_{\gimel} \beth^{\gamma}{}_{F^{\gamma}} \right) \right)$$

By Theorem 3.1,

$$\begin{aligned} &\text{LDFWA}(\varsigma^{\vartheta}{}_{1} \oplus F^{\gamma}, \varsigma^{\vartheta}{}_{2} \oplus F^{\gamma}, \ldots \varsigma^{\vartheta}{}_{n} \oplus F^{\gamma}) \\ &= \left(\left\langle (1 - \overline{\prod}_{\gimel=1}^{n} \left((1 - \zeta^{\tau}{}_{\gimel})(1 - \zeta^{\tau}{}_{F^{\gamma}}) \right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \left(\daleth^{\upsilon}{}_{F^{\gamma}} \daleth^{\upsilon}{}_{\gimel} \right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right. \\ &\quad \left. \left\langle (1 - \overline{\prod}_{\gimel=1}^{n} \left((1 - \xi^{\hbar}{}_{\gimel})(1 - \xi^{\hbar}{}_{F^{\gamma}}) \right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \left(\beth^{\gamma}{}_{F^{\gamma}} \beth^{\gamma}{}_{\gimel} \right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right) \\ &= \left(\left\langle (1 - \left(1 - \zeta^{\tau}{}_{F^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(1 - \zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\daleth^{\upsilon}{}_{F^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(\daleth^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right. \\ &\quad \left. \left\langle (1 - \left(1 - \xi^{\hbar}{}_{F^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(1 - \xi^{\hbar}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\beth^{\gamma}{}_{F^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(\beth^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right) \\ &= \left(\left\langle (1 - \left(1 - \zeta^{\tau}{}_{F^{\gamma}}\right) \overline{\prod}_{\gimel=1}^{n} \left(1 - \zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\daleth^{\upsilon}{}_{F^{\gamma}}\right) \overline{\prod}_{\gimel=1}^{n} \left(\daleth^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right. \\ &\quad \left. \left\langle (1 - \left(1 - \xi^{\hbar}{}_{F^{\gamma}}\right) \overline{\prod}_{\gimel=1}^{n} \left(1 - \xi^{\hbar}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\beth^{\gamma}{}_{F^{\gamma}}\right) \overline{\prod}_{\gimel=1}^{n} \left(\beth^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right) \end{aligned}$$

Now, by operational laws of LDFNs,

$$\begin{aligned} &\text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{\mathrm{n}}) \oplus F^{\gamma} \\ &= \left(\left\langle (1 - \overline{\prod}_{\gimel=1}^{n} (1 - \zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \daleth^{\upsilon \mathfrak{P}^{\gamma}{}_{\gimel}}_{\gimel} \right\rangle, \left\langle (1 - \overline{\prod}_{\gimel=1}^{n} (1 - \xi^{\hbar}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \beth^{\gamma \mathfrak{P}^{\gamma}{}_{\gimel}}_{\gimel} \right\rangle \oplus \right. \\ &\quad \left. (\langle \zeta^{\tau}{}_{F^{\gamma}}, \daleth^{\upsilon}{}_{F^{\gamma}} \rangle, \langle \xi^{\hbar}{}_{F^{\gamma}}, \beth^{\gamma}{}_{F^{\gamma}} \rangle) \right) \\ &= \left(\left\langle (1 - \left(1 - \zeta^{\tau}{}_{F^{\gamma}}\right) \overline{\prod}_{\gimel=1}^{n} \left(1 - \zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\daleth^{\upsilon}{}_{F^{\gamma}}\right) \overline{\prod}_{\gimel=1}^{n} \left(\daleth^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right. \\ &\quad \left. \left\langle (1 - \left(1 - \xi^{\hbar}{}_{F^{\gamma}}\right) \overline{\prod}_{\gimel=1}^{n} \left(1 - \xi^{\hbar}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\beth^{\gamma}{}_{F^{\gamma}}\right) \overline{\prod}_{\gimel=1}^{n} \left(\beth^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right) \end{aligned}$$

Thus,

$$\text{LDFWA}(\varsigma^{\vartheta}{}_{1} \oplus F^{\gamma}, \varsigma^{\vartheta}{}_{2} \oplus F^{\gamma}, \ldots \varsigma^{\vartheta}{}_{\mathrm{n}} \oplus F^{\gamma}) = \text{LDFWA}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{\mathrm{n}}) \oplus F^{\gamma}$$

3.
According to Theorem 3.1,

$$\begin{aligned} &\text{q-ROFWA}(\varsigma^{\vartheta}{}_{1} \oplus F^{\gamma}{}_{2}, \varsigma^{\vartheta}{}_{2} \oplus F^{\gamma}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n} \oplus F^{\gamma}{}_{n}) \\ &= \left(\left\langle 1 - \overline{\prod}_{\gimel=1}^{n} \left((1 - \zeta^{\tau}{}_{\gimel})(1 - \phi_{\gimel}) \right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \left(\varphi_{\gimel} \daleth^{\upsilon}{}_{\gimel} \right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right. \\ &\quad \left. \left\langle 1 - \overline{\prod}_{\gimel=1}^{n} \left((1 - \xi^{\hbar}{}_{\gimel})(1 - \mathscr{K}_{\gimel}) \right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \left(\mathscr{M}_{\gimel} \beth^{\gamma}{}_{\gimel} \right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right) \\ &= \left(\left\langle 1 - \overline{\prod}_{\gimel=1}^{n} \left(1 - \phi_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(1 - \zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \left(\varphi_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(\daleth^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right. \\ &\quad \left. \left\langle 1 - \overline{\prod}_{\gimel=1}^{n} \left(1 - \mathscr{K}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(1 - \xi^{\hbar}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \left(\mathscr{M}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(\beth^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right) \end{aligned}$$

Now,

$$
\begin{aligned}
&\mathrm{LDFWA}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \ldots \varsigma^{\vartheta}{}_n) \oplus \mathrm{LDFWA}(F^{\gamma}{}_1, F^{\gamma}{}_2, \ldots F^{\gamma}{}_n) \\
&= \left(\left\langle 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \daleth^{\upsilon}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \xi^{\hbar}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \beth^{\gamma}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right) \oplus \\
&\left(\left\langle 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \phi_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \varphi_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \mathscr{K}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \mathscr{M}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right) \\
&= \left(\left\langle 1 - \overline{\prod}_{\gimel=1}^{n} \left(1 - \phi_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(1 - \zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \left(\varphi_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(\daleth^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle,\right. \\
&\left.\left\langle 1 - \overline{\prod}_{\gimel=1}^{n} \left(1 - \mathscr{K}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(1 - \xi^{\hbar}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \left(\mathscr{M}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \overline{\prod}_{\gimel=1}^{n} \left(\beth^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right)
\end{aligned}
$$

Thus,

$$
\mathrm{LDFWA}(\varsigma^{\vartheta}{}_1 \oplus F^{\gamma}{}_2, \varsigma^{\vartheta}{}_2 \oplus F^{\gamma}{}_2, \ldots \varsigma^{\vartheta}{}_{\mathrm{n}} \oplus F^{\gamma}{}_{\mathrm{n}}) = \mathrm{LDFWA}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \ldots \varsigma^{\vartheta}{}_{\mathrm{n}}) \oplus \mathrm{LDFWA}(F^{\gamma}{}_1, F^{\gamma}{}_2, \ldots F^{\gamma}{}_{\mathrm{n}})
$$

□

3.2. LDFWG operator

Definition 3.2. Consider $\varsigma^{\vartheta}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \daleth^{\upsilon}{}_{\gimel} \rangle, \langle \xi^{\hbar}{}_{\gimel}, \beth^{\gamma}{}_{\gimel} \rangle)$ is the collection of LDFNs and $LDFWG : \Bbbk^{n} \to \Bbbk$ be the mapping.

$$
\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \ldots \varsigma^{\vartheta}{}_n) = \varsigma^{\vartheta}{}_1^{\mathfrak{P}^{\gamma}{}_1} \otimes \varsigma^{\vartheta}{}_2^{\mathfrak{P}^{\gamma}{}_2} \otimes \ldots, \otimes \varsigma^{\vartheta}{}_n^{\mathfrak{P}^{\gamma}{}_n} \quad (3)
$$

then the mapping LDFWG is called LDFWG operator, where $(\mathfrak{P}^{\gamma}{}_1, \mathfrak{P}^{\gamma}{}_2, \ldots, \mathfrak{P}^{\gamma}{}_n)$ be the WV with the constraint $\mathfrak{P}^{\gamma}{}_i > 0$ and $\sum_{i=1}^{n} \mathfrak{P}^{\gamma}{}_i = 1$.

We might also think about LDFWG operator by employing the theorem following.

Theorem 3.5. Assume that $\varsigma^{\vartheta}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \daleth^{\upsilon}{}_{\gimel} \rangle, \langle \xi^{\hbar}{}_{\gimel}, \beth^{\gamma}{}_{\gimel} \rangle)$ is the collection of LDFNs, we also evaluate LDFWG by

$$
\begin{aligned}
&\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \ldots \varsigma^{\vartheta}{}_n) \\
&= \left(\left\langle \overline{\prod}_{\gimel=1}^{n} \zeta^{\tau}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle \overline{\prod}_{\gimel=1}^{n} \xi^{\hbar}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right)
\end{aligned} \quad (4)
$$

Proof. It is quite simple for the first assertion to come before the Definition 3.2 and the Theorem 3.5. The following instances demonstrate this point further:

$$
\begin{aligned}
&\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \ldots \varsigma^{\vartheta}{}_n) = \varsigma^{\vartheta}{}_1^{\mathfrak{P}^{\gamma}{}_1} \otimes \varsigma^{\vartheta}{}_2^{\mathfrak{P}^{\gamma}{}_2} \otimes \ldots, \otimes \varsigma^{\vartheta}{}_n^{\mathfrak{P}^{\gamma}{}_n} \\
&= \left(\left\langle \overline{\prod}_{\gimel=1}^{n} \zeta^{\tau}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle \overline{\prod}_{\gimel=1}^{n} \xi^{\hbar}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right)
\end{aligned}
$$

In order to demonstrate the validity of this theorem, we turned to mathematics induction.
For $n = 2$

$$
\varsigma^{\vartheta}{}_1^{\mathfrak{P}^{\gamma}{}_1} = \left(\left\langle \zeta^{\tau}{}_1^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \daleth^{\upsilon}{}_1)^{\mathfrak{P}^{\gamma}{}_1} \right\rangle, \left\langle \xi^{\hbar}{}_1^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \beth^{\gamma}{}_1)^{\mathfrak{P}^{\gamma}{}_1} \right\rangle\right)
$$

$$
\varsigma^{\vartheta}{}_2^{\mathfrak{P}^{\gamma}{}_2} = \left(\left\langle \zeta^{\tau}{}_2^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1} \right\rangle, \left\langle \xi^{\hbar}{}_2^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1} \right\rangle\right)
$$

Then,

$$
\begin{aligned}
&\varsigma^{\vartheta}{}_1^{\mathfrak{P}^{\gamma}{}_1} \otimes \varsigma^{\vartheta}{}_2^{\mathfrak{P}^{\gamma}{}_2} \\
&= \left(\left\langle \zeta^{\tau}{}_1^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \daleth^{\upsilon}{}_1)^{\mathfrak{P}^{\gamma}{}_1} \right\rangle, \left\langle \xi^{\hbar}{}_1^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \beth^{\gamma}{}_1)^{\mathfrak{P}^{\gamma}{}_1} \right\rangle\right) \otimes \left(\left\langle \zeta^{\tau}{}_2^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1} \right\rangle, \left\langle \xi^{\hbar}{}_2^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1} \right\rangle\right) \\
&= \left(\left\langle \zeta^{\tau}{}_1^{\mathfrak{P}^{\gamma}{}_1} . \zeta^{\tau}{}_2^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \daleth^{\upsilon}{}_1)^{\mathfrak{P}^{\gamma}{}_1} + 1 - (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1} - \left(1 - (1 - \daleth^{\upsilon}{}_1)^{\mathfrak{P}^{\gamma}{}_1}\right)\left(1 - (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1}\right) \right\rangle, \left\langle \xi^{\hbar}{}_1^{\mathfrak{P}^{\gamma}{}_1} . \xi^{\hbar}{}_2^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \beth^{\gamma}{}_1)^{\mathfrak{P}^{\gamma}{}_1} + 1 - (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1} - \right.\right. \\
&\left.\left.\left(1 - (1 - \beth^{\gamma}{}_1)^{\mathfrak{P}^{\gamma}{}_1}\right)\left(1 - (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1}\right) \right\rangle\right) \\
&= \left(\left\langle \zeta^{\tau}{}_1^{\mathfrak{P}^{\gamma}{}_1} . \zeta^{\tau}{}_2^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \daleth^{\upsilon}{}_1)^{\mathfrak{P}^{\gamma}{}_1} (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1} \right\rangle, \left\langle \xi^{\hbar}{}_1^{\mathfrak{P}^{\gamma}{}_1} . \xi^{\hbar}{}_2^{\mathfrak{P}^{\gamma}{}_1}, 1 - (1 - \beth^{\gamma}{}_1)^{\mathfrak{P}^{\gamma}{}_1} (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_1} \right\rangle\right) \\
&= \left(\left\langle \overline{\prod}_{\gimel=1}^{2} \zeta^{\tau}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \overline{\prod}_{\gimel=1}^{2} (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle \overline{\prod}_{\gimel=1}^{2} \xi^{\hbar}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \overline{\prod}_{\gimel=1}^{2} (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right)
\end{aligned}
$$

This shows that Equation 4 is true for $n = 2$, and now assume that Equation 4 holds for $n = k$, that is,

$$\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \dots \varsigma^{\vartheta}{}_k) = \left(\left\langle \prod_{\gimel=1}^{k} \zeta^{\tau}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \prod_{\gimel=1}^{k}(1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle \prod_{\gimel=1}^{k} \xi^{\hbar}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \prod_{\gimel=1}^{k}(1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right)$$

Now $n = k + 1$, by operational laws of LDFNs we have

$$\begin{aligned}
\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \dots \varsigma^{\vartheta}{}_{k+1}) &= \mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \dots \varsigma^{\vartheta}{}_k) \otimes \varsigma^{\vartheta}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \\
&= \left(\left\langle \prod_{\gimel=1}^{k} \zeta^{\tau}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \prod_{\gimel=1}^{k}(1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle \prod_{\gimel=1}^{k} \xi^{\hbar}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \prod_{\gimel=1}^{k}(1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right) \otimes \\
&\quad \left(\left\langle \zeta^{\tau}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}, 1 - (1 - \daleth^{\upsilon}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}} \right\rangle, \left\langle \xi^{\hbar}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}, 1 - (1 - \beth^{\gamma}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}} \right\rangle\right) \\
&= \left(\left\langle \prod_{\gimel=1}^{k} \zeta^{\tau}{}_{k}^{\mathfrak{P}^{\gamma}{}_{\gimel}} . \zeta^{\tau}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}, 1 - \prod_{\gimel=1}^{k}(1 - \daleth^{\upsilon}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\gimel}}(1 - \daleth^{\upsilon}{}_{k+1})^{k+1} \right\rangle, \right. \\
&\quad \left. \left\langle \prod_{\gimel=1}^{k} \xi^{\hbar}{}_{k}^{\mathfrak{P}^{\gamma}{}_{\gimel}} . \xi^{\hbar}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}, 1 - \prod_{\gimel=1}^{k}(1 - \beth^{\gamma}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\gimel}}(1 - \beth^{\gamma}{}_{k+1})^{k+1} \right\rangle\right) \\
&= \left(\left\langle \prod_{\gimel=1}^{k+1} \zeta^{\tau}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \prod_{\gimel=1}^{k+1}(1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle \prod_{\gimel=1}^{k+1} \xi^{\hbar}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \prod_{\gimel=1}^{k+1}(1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right)
\end{aligned}$$

This shows that for $n = k + 1$, Equation 2 holds. Then,

$$\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \dots \varsigma^{\vartheta}{}_n) = \left(\left\langle \prod_{\gimel=1}^{n} \zeta^{\tau}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \prod_{\gimel=1}^{n}(1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \left\langle \prod_{\gimel=1}^{n} \xi^{\hbar}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}, 1 - \prod_{\gimel=1}^{n}(1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle\right)$$

□

Theorem 3.6. Assume that $\varsigma^{\vartheta}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \daleth^{\upsilon}{}_{\gimel}\rangle, \langle \xi^{\hbar}{}_{\gimel}, \beth^{\gamma}{}_{\gimel}\rangle)$ is the collection of LDFNs. If all $\varsigma^{\vartheta}{}_{\gimel}$ are equal, that is, $\varsigma^{\vartheta}{}_{\gimel} = \varsigma^{\vartheta}$ $\forall j$, then

$$\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \dots \varsigma^{\vartheta}{}_{\mathrm{n}}) = \varsigma^{\vartheta}$$

Proof. From Definition 3.1, we have

$$\begin{aligned}
\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \dots \varsigma^{\vartheta}{}_n) &= \varsigma^{\vartheta}{}_1^{\mathfrak{P}^{\gamma}{}_1} \otimes \varsigma^{\vartheta}{}_2^{\mathfrak{P}^{\gamma}{}_2} \otimes \dots, \otimes \varsigma^{\vartheta}{}_n^{\mathfrak{P}^{\gamma}{}_n} \\
&= \varsigma^{\vartheta \mathfrak{P}^{\gamma}{}_1} \otimes \varsigma^{\vartheta \mathfrak{P}^{\gamma}{}_2} \otimes \dots, \otimes \varsigma^{\vartheta \mathfrak{P}^{\gamma}{}_n} \\
&= \varsigma^{\vartheta}
\end{aligned}$$

□

Corollary 3.2. If $\varsigma^{\vartheta}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \daleth^{\upsilon}{}_{\gimel}\rangle, \langle \xi^{\hbar}{}_{\gimel}, \beth^{\gamma}{}_{\gimel}\rangle)$ is the collection of absolute LDFNs, that is, $\varsigma^{\vartheta}{}_{\gimel} = (\langle 1,0\rangle, \langle 1,0\rangle)$ for all j, then

$$\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \dots \varsigma^{\vartheta}{}_{\mathrm{n}}) = (\langle 1, 0\rangle, \langle 1, 0\rangle)$$

Proof. We can easily obtain Corollary similar to the Theorem 3.2. □

Theorem 3.7. Assume that $\varsigma^{\vartheta}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \daleth^{\upsilon}{}_{\gimel}\rangle, \langle \xi^{\hbar}{}_{\gimel}, \beth^{\gamma}{}_{\gimel}\rangle)$ and $\varsigma^{\vartheta}{}_{\gimel}{}^{*} = (\langle \zeta^{\tau *}{}_{\gimel}, \daleth^{\upsilon *}{}_{\gimel}\rangle, \langle \xi^{\hbar *}{}_{\gimel}, \beth^{\gamma *}{}_{\gimel}\rangle)$ are the assemblages of LDFNs. If $\zeta^{\tau *}{}_{\gimel} \geq \zeta^{\tau}{}_{\gimel}$, $\daleth^{\upsilon *}{}_{\gimel} \leq \daleth^{\upsilon}{}_{\gimel}$, $\xi^{\hbar *}{}_{\gimel} \geq \xi^{\hbar}{}_{\gimel}$ and $\beth^{\gamma *}{}_{\gimel} \leq \beth^{\gamma}{}_{\gimel}$ for all j, then

$$\mathrm{LDFWG}(\varsigma^{\vartheta}{}_1, \varsigma^{\vartheta}{}_2, \dots \varsigma^{\vartheta}{}_{\mathrm{n}}) \leq \mathrm{LDFWG}(\varsigma^{\vartheta *}_1, \varsigma^{\vartheta *}_2, \dots \varsigma^{\vartheta *}_{\mathrm{n}})$$

Proof. Here, $\daleth^{\upsilon *}{}_{\gimel} \geq \daleth^{\upsilon}{}_{\gimel}$ and $\zeta^{\tau *}{}_{\gimel} \leq \zeta^{\tau}{}_{\gimel}$ for all j, If $\daleth^{\upsilon *}{}_{\gimel} \geq \daleth^{\upsilon}{}_{\gimel}$.

$$\Leftrightarrow \daleth^{\upsilon *}{}_{\gimel} \geq \daleth^{\upsilon}{}_{\gimel} \Leftrightarrow 1 - \daleth^{\upsilon *}{}_{\gimel} \leq 1 - \daleth^{\upsilon}{}_{\gimel}$$

$$\Leftrightarrow (1 - \daleth^{\upsilon *}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

$$\Leftrightarrow \prod_{\gimel=1}^{n}(1 - \daleth^{\upsilon *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \prod_{\gimel=1}^{n}(1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

$$\Leftrightarrow 1 - \prod_{\gimel=1}^{n}(1 - \daleth^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq 1 - \prod_{\gimel=1}^{n}(1 - \daleth^{\upsilon *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

And,
$\beth^{\gamma *}{}_{\gimel} \geq \beth^{\gamma}{}_{\gimel}$ and $\xi^{\hbar *}{}_{\gimel} \leq \xi^{\hbar}{}_{\gimel}$ for all j, If $\beth^{\gamma *}{}_{\gimel} \geq \beth^{\gamma}{}_{\gimel}$.

$$\Leftrightarrow \beth^{\gamma *}{}_{\gimel} \geq \beth^{\gamma}{}_{\gimel} \Leftrightarrow 1 - \beth^{\gamma *}{}_{\gimel} \leq 1 - \beth^{\gamma}{}_{\gimel}$$

$$\Leftrightarrow (1 - \beth^{\gamma *}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

$$\Leftrightarrow \prod_{\gimel=1}^{n}(1 - \beth^{\gamma *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \prod_{\gimel=1}^{n}(1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

$$\Leftrightarrow 1 - \prod_{\gimel=1}^{n}(1 - \beth^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq 1 - \prod_{\gimel=1}^{n}(1 - \beth^{\gamma *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

Now,

$\zeta^{\tau *}{}_{\gimel} \leq \zeta^{\tau}{}_{\gimel}$.

$$\Leftrightarrow (\zeta^{\tau *}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (\zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \Leftrightarrow \prod_{\gimel=1}^{n}(\zeta^{\tau *}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \prod_{\gimel=1}^{n}(\zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$$

And,
$\xi^{\hbar *}{}_{\gimel} \leq \xi^{\hbar}{}_{\gimel}$.

$\Leftrightarrow (\xi^{\hbar *}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (\xi^{\hbar}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$

$\Leftrightarrow \overline{\prod_{\gimel=1}^{n}} (\xi^{\hbar *}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod_{\gimel=1}^{n}} (\xi^{\hbar}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$

Let

$$\overline{\varsigma^{\vartheta}} = \text{LDFWG}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{\text{n}})$$

and

$$\overline{\varsigma^{\vartheta *}} = \text{LDFWG}(\varsigma^{\vartheta *}_{1}, \varsigma^{\vartheta *}_{2}, \ldots \varsigma^{\vartheta *}_{\text{n}})$$

We get that $\overline{\varsigma^{\vartheta *}} \geq \overline{\varsigma^{\vartheta}}$. So,

$$\text{LDFWG}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{\text{n}}) \leq \text{LDFWG}(\varsigma^{\vartheta *}_{1}, \varsigma^{\vartheta *}_{2}, \ldots \varsigma^{\vartheta *}_{\text{n}})$$

□

Theorem 3.8. Assume that $\varsigma^{\vartheta}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \daleth^{\upsilon}{}_{\gimel}\rangle, \langle \xi^{\hbar}{}_{\gimel}, \beth^{\gamma}{}_{\gimel}\rangle)$ and $\digamma^{\gamma}{}_{\gimel} = (\langle \varphi_{\gimel}, \phi_{\gimel}\rangle, \langle \mathscr{K}_{\gimel}, \mathscr{M}_{\gimel}\rangle)$ are two families of LDFNs. If $r > 0$ and $\digamma^{\gamma} = (\langle \zeta^{\tau}{}_{\digamma^{\gamma}}, \daleth^{\upsilon}{}_{\digamma^{\gamma}}\rangle, \langle \xi^{\hbar}{}_{\digamma^{\gamma}}, \beth^{\gamma}{}_{\digamma^{\gamma}}\rangle)$ is a LDFN, then

1. $\text{LDFWG}(\varsigma^{\vartheta}{}_{1} \oplus \digamma^{\gamma}, \varsigma^{\vartheta}{}_{2} \oplus \digamma^{\gamma}, \ldots \varsigma^{\vartheta}{}_{n} \oplus \digamma^{\gamma}) = \text{LDFWG}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n}) \oplus \digamma^{\gamma}$
2. $\text{LDFWG}(r\varsigma^{\vartheta}{}_{1}, r\varsigma^{\vartheta}{}_{2}, \ldots r\varsigma^{\vartheta}{}_{n}) = r\ \text{LDFWG}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n})$
3. $\text{LDFWG}(\varsigma^{\vartheta}{}_{1} \oplus \digamma^{\gamma}{}_{1}, \varsigma^{\vartheta}{}_{2} \oplus \digamma^{\gamma}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n} \oplus \digamma^{\gamma}{}_{n}) = \text{LDFWG}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n}) \oplus \text{LDFWG}(\digamma^{\gamma}{}_{1}, \digamma^{\gamma}{}_{2}, \ldots \digamma^{\gamma}{}_{n})$
4. $\text{LDFWG}(r\varsigma^{\vartheta}{}_{1} \oplus \digamma^{\gamma}, r\varsigma^{\vartheta}{}_{2} \oplus \digamma^{\gamma}, \ldots \oplus r\varsigma^{\vartheta}{}_{n} \oplus \digamma^{\gamma}) = r\ \text{LDFWG}(\varsigma^{\vartheta}{}_{1}, \varsigma^{\vartheta}{}_{2}, \ldots \varsigma^{\vartheta}{}_{n}) \oplus \digamma^{\gamma}$

Proof. The proof of this theorem is same as Theorem 3.4. □

4. Proposed Methodology Based on Developed AOs

Let $\mathscr{T}^{\gimel} = \{\mathscr{T}^{\gimel}_{1}, \mathscr{T}^{\gimel}_{2}, \ldots, \mathscr{T}^{\gimel}_{m}\}$ and $\breve{\mathscr{G}}^{\zeta} = \{\breve{\mathscr{G}}^{\zeta}{}_{1}, \breve{\mathscr{G}}^{\zeta}{}_{2}, \ldots, \breve{\mathscr{G}}^{\zeta}{}_{n}\}$ are the alternatives and criterion, respectively. DM offered his judgment matrix $D = (\aleph^{\Bbbk}_{ij})_{m \times n}$, in which $\aleph^{\Bbbk}_{ij}$ stands for the alternate $\mathscr{T}^{\gimel}_{i} \in \mathscr{T}^{\gimel}$ as per the parameter $\breve{\mathscr{G}}^{\zeta}{}_{\gimel} \in \breve{\mathscr{G}}^{\zeta}$ by DM. The matrix D has converted into "normalized matrix" by the given formula "$Y = (\varsigma^{\vartheta\wp}{}_{ij})_{m \times n}$",

$$(\varsigma^{\vartheta\wp}{}_{ij})_{m \times n} = \begin{cases} (\aleph_{ij})^{c}; & j \in \tau_{c} \\ \aleph_{ij}; & j \in \tau_{b} \end{cases}. \tag{5}$$

where $(\aleph^{\Bbbk}_{ij})^{c}$ denotes the compliment of $\aleph^{\Bbbk}_{ij}$.

The MCDM will be updated to include the suggested operators, which will make the previously described processes necessary.

Algorithm

Step 1: Acquire the judgment matrix $D = (\aleph^{\Bbbk}_{ij})_{m \times n}$ based on LDFNs from DMs.

$$\begin{array}{cc} & \begin{array}{cccc} \breve{\mathscr{G}}_{1} & \breve{\mathscr{G}}_{2} & & \breve{\mathscr{G}}_{n} \end{array} \\ \begin{array}{c} \breve{\mathscr{G}}_{1} \\ \breve{\mathscr{G}}_{2} \\ \\ \breve{\mathscr{G}}_{n} \end{array} & \begin{bmatrix} (\langle \zeta^{\tau}{}_{11}, \daleth^{\upsilon}{}_{11}\rangle, \langle \xi^{\hbar}{}_{11}, \beth^{\gamma}{}_{11}\rangle) & (\langle \zeta^{\tau}{}_{12}, \daleth^{\upsilon}{}_{12}\rangle, \langle \xi^{\hbar}{}_{12}, \beth^{\gamma}{}_{12}\rangle) & \cdots\cdots & (\langle \zeta^{\tau}{}_{1n}, \daleth^{\upsilon}{}_{1n}\rangle, \langle \xi^{\hbar}{}_{1n}, \beth^{\gamma}{}_{1n}\rangle) \\ (\langle \zeta^{\tau}{}_{21}, \daleth^{\upsilon}{}_{21}\rangle, \langle \xi^{\hbar}{}_{21}, \beth^{\gamma}{}_{21}\rangle) & (\langle \zeta^{\tau}{}_{22}, \daleth^{\upsilon}{}_{22}\rangle, \langle \xi^{\hbar}{}_{22}, \beth^{\gamma}{}_{22}\rangle) & \cdots\cdots & (\langle \zeta^{\tau}{}_{2n}, \daleth^{\upsilon}{}_{2n}\rangle, \langle \xi^{\hbar}{}_{2n}, \beth^{\gamma}{}_{2n}\rangle) \\ \vdots & \vdots & \ddots\ \ddots & \vdots \\ (\langle \zeta^{\tau}{}_{m1}, \daleth^{\upsilon}{}_{m1}\rangle, \langle \xi^{\hbar}{}_{m1}, \beth^{\gamma}{}_{m1}\rangle) & (\langle \zeta^{\tau}{}_{m2}, \daleth^{\upsilon}{}_{m2}\rangle, \langle \xi^{\hbar}{}_{m2}, \beth^{\gamma}{}_{m2}\rangle) & \cdots\cdots & (\langle \zeta^{\tau}{}_{mn}, \daleth^{\upsilon}{}_{mn}\rangle, \langle \xi^{\hbar}{}_{mn}, \beth^{\gamma}{}_{mn}\rangle) \end{bmatrix} \end{array}$$

Step 2: There is no need for normalization if all indicators are of the same kind. The matrix D has amended to "transforming response matrix, $Y = (\varsigma^{\vartheta}{}_{ij}{}^{\wp})_{m \times n}$" by Equation 5.

Step 3: Aggregate $\mathscr{R}^{S}_{ij}$ for all alternates $\mathscr{T}^{\gimel}{}_{i}$ by utilizing the LDFWA (LDFWG) operator.

$$\mathscr{R}^{S}{}_{ij} = LDFWA(\varsigma^{\vartheta\wp}_{i1}, \varsigma^{\vartheta\wp}_{i2}, \ldots \varsigma^{\vartheta\wp}_{in})$$

or

$$\mathscr{R}^{S}{}_{ij} = LDFWG(\varsigma^{\vartheta\wp}_{i1}, \varsigma^{\vartheta\wp}_{i2}, \ldots \varsigma^{\vartheta\wp}_{in})$$

Step 4: Compute the score against all the alternatives.

Step 5: The SF was used to classify the alternatives, and the most appropriate option was chosen.

5. MCDM Example

MCDM is a method used to evaluate and select among multiple options, taking into account multiple criteria or factors that are important to the DM. In the field of agriculture, MCDM can be used to help farmers, researchers, and policymakers make more informed decisions about crop selection, land use, and other important agricultural activities. One important application of MCDM in agriculture is in crop selection. When choosing which crops to plant, farmers need to consider factors such as the climate, soil type, water availability, market demand, and potential yields. By using MCDM, farmers can evaluate multiple options and select the one that best meets their needs and goals. For example, a farmer might use MCDM to compare the yield potential and water requirements of different varieties of wheat and select the one that offers the best balance between these two factors.

Another important application of MCDM in agriculture is in land use planning. When deciding how to use land, policymakers and researchers need to consider factors such as the potential for crop production, the impact of different land uses on the environment, and the social and economic benefits of different land uses. MCDM can help DMs evaluate different options and identify the one that offers the best overall balance of these factors. MCDM can also be used to support sustainable agricultural practices by assessing and prioritizing the different ecological and socioeconomic aspects of a system. It could also assist to evaluate the tradeoffs and benefits of different management practices and support technology/innovation adoption.

In Pakistan, agriculture is a major contributor to the economy and a source of livelihood for many people. However, the country faces several challenges in this sector, including water scarcity, land degradation, and the impact of climate change. By using MCDM, DMs in Pakistan can work to address these challenges and promote sustainable agricultural practices that benefit both farmers and the environment. MCDM methods allow DMs to take into account multiple criteria and provide a transparent, systematic way to evaluate different options. Overall, MCDM can be a valuable tool for farmers, researchers, and policymakers in the field of agriculture, particularly in Pakistan, as it allows for comprehensive and systematic evaluations of different options and their tradeoffs, based on the criteria that are important to the DMs.

Agriculture is a significant contributor to Pakistan's economy, accounting for 18.9 percent of the country's gross domestic product and employing 42.3 percent of the labor force. In addition to this, it is a significant source of revenues from international commerce and it encourages growth in a variety of other areas. To boost development in this field, the public authority is focusing on aiding small and marginalized ranchers and pushing limited scope creative solutions. The sixth population and housing census that was conducted in Pakistan in 2017 revealed that the country's overall population is expanding at a pace of 2.4 percent on an annual basis. Demand for goods produced by agriculture is expected to rise as a result of the fast population expansion. The current administration is centered on advancing this area and has begun various measures, for example, crop expansion, decreasing increase rates, proficient utilization of water, and advancement of high worth yields including biotechnology, agribusiness credit advancement, subsidized manure costs, and modest power for negritude wells. As a result, this current area's exhibition expanded complicated after undergoing moderate and slowed expansion over the previous 13 years.

Considering the decision-making challenge of determining the best agricultural land. Assume the collection of choices, $\mathscr{T}^{\lambda}_1$, $\mathscr{T}^{\lambda}_2$, $\mathscr{T}^{\lambda}_3$ and $\mathscr{T}^{\lambda}_4$, also considering four criterions, $\wp^{\mathfrak{R}}{}_1$ = irrigation, $\wp^{\mathfrak{R}}{}_2$ = cost, $\wp^{\mathfrak{R}}{}_3$ = soil and $\wp^{\mathfrak{R}}{}_4$ = processing industry and market. Assuming that the criteria were weighted as (0.25,0.40,0.20,0.15).

Algorithm

5.1. With LDFWA operator

Step 1: Obtain matrix $D = (\aleph^{\Bbbk}_{ij})_{m\times n}$ by DM, which is shown in Table 1.

Step 2: In this case, $\check{\mathscr{G}}^{\zeta}{}_2$ criteria is cost type criteria, and all are the benefits types, so there is need of normalization. Normalized LDF decision matrix is given in Table 2.

Step 3: Aggregate the LDF values $\mathscr{R}^{S}{}_{ij}$ for all $\mathscr{T}^{\lambda}_i$ using LDFWA operator, given in Table 3.

Table 3
LDF-aggregated values $\mathscr{R}^{S}{}_i$

$\mathscr{R}^{S}{}_1$	(⟨0.596248,0.760098⟩,⟨0.32997,0.175855⟩)
$\mathscr{R}^{S}{}_2$	(⟨0.769462,0.522578⟩,⟨0.523542,0.612701⟩)
$\mathscr{R}^{S}{}_3$	(⟨0.503278,0.624946⟩,⟨0.708147,0.613116⟩)
$\mathscr{R}^{S}{}_4$	(⟨0.482460,0.581847⟩,⟨0.532108,0.399725⟩)

Step 4: Compute the score for all LDF-aggregated values $\mathscr{R}^{S}_i$.

$$\beth^{\lambda}(\mathscr{R}^{S}{}_1) = 0.497566$$

$$\beth^{\lambda}(\mathscr{R}^{S}{}_2) = 0.539431$$

$$\beth^{\lambda}(\mathscr{R}^{S}{}_3) = 0.493341$$

$$\beth^{\lambda}(\mathscr{R}^{S}{}_4) = 0.508249$$

Step 5: Ranks according to SFs.

$$\mathscr{R}^{S}{}_2 \succ \mathscr{R}^{S}{}_4 \succ \mathscr{R}^{S}{}_1 \succ \mathscr{R}^{S}{}_3$$

So,

$$\mathscr{T}^{\lambda}_2 \succ \mathscr{T}^{\lambda}_1 \succ \mathscr{T}^{\lambda}{}_4 \succ \mathscr{T}^{\lambda}_3$$

$\mathscr{T}^{\lambda}_2$ is best alternative among all other alternatives.

Table 1
Rating given by DM

	$\check{\mathscr{G}}^{\zeta}_1$	$\check{\mathscr{G}}^{\zeta}{}_2$	$\check{\mathscr{G}}^{\zeta}{}_3$	$\check{\mathscr{G}}^{\zeta}{}_4$
$\mathscr{T}^{\lambda}_1$	(⟨0.50,0.85⟩,⟨0.30,0.10⟩)	(⟨0.45,0.70⟩,⟨0.25,0.20⟩)	(⟨0.65,0.75⟩,⟨0.45,0.25⟩)	(⟨0.85,0.80⟩,⟨0.40,0.20⟩)
$\mathscr{T}^{\lambda}_2$	(⟨0.80,0.90⟩,⟨0.45,0.15⟩)	(⟨0.45,0.65⟩,⟨0.55,0.35⟩)	(⟨0.75,0.45⟩,⟨0.40,0.30⟩)	(⟨0.65,0.85⟩,⟨0.45,0.35⟩)
$\mathscr{T}^{\lambda}_3$	(⟨0.35,0.65⟩,⟨0.50,0.20⟩)	(⟨0.65,0.95⟩,⟨0.25,0.65⟩)	(⟨0.45,0.90⟩,⟨0.30,0.45⟩)	(⟨0.55,0.95⟩,⟨0.50,0.30⟩)
$\mathscr{T}^{\lambda}_4$	(⟨0.50,0.50⟩,⟨0.50,0.25⟩)	(⟨0.90,0.55⟩,⟨0.50,0.40⟩)	(⟨0.45,0.65⟩,⟨0.35,0.50⟩)	(⟨0.35,0.65⟩,⟨0.30,0.20⟩)

Table 2
Normalized LDF decision matrix

	$\check{\mathscr{G}}^{\zeta}_1$	$\check{\mathscr{G}}^{\zeta}{}_2$	$\check{\mathscr{G}}^{\zeta}{}_3$	$\check{\mathscr{G}}^{\zeta}{}_4$
$\mathscr{T}^{\lambda}_1$	(⟨0.50,0.85⟩,⟨0.30,0.10⟩)	(⟨0.70,0.45⟩,⟨0.20,0.25⟩)	(⟨0.65,0.75⟩,⟨0.45,0.25⟩)	(⟨0.85,0.80⟩,⟨0.40,0.20⟩)
$\mathscr{T}^{\lambda}_2$	(⟨0.80,0.90⟩,⟨0.45,0.15⟩)	(⟨0.65,0.45⟩,⟨0.35,0.55⟩)	(⟨0.75,0.45⟩,⟨0.40,0.30⟩)	(⟨0.65,0.85⟩,⟨0.45,0.35⟩)
$\mathscr{T}^{\lambda}_3$	(⟨0.35,0.65⟩,⟨0.50,0.20⟩)	(⟨0.95,0.65⟩,⟨0.65,0.25⟩)	(⟨0.45,0.90⟩,⟨0.30,0.45⟩)	(⟨0.55,0.95⟩,⟨0.50,0.30⟩)
$\mathscr{T}^{\lambda}_4$	(⟨0.50,0.50⟩,⟨0.50,0.25⟩)	(⟨0.55,0.90⟩,⟨0.40,0.50⟩)	(⟨0.45,0.65⟩,⟨0.35,0.50⟩)	(⟨0.35,0.65⟩,⟨0.30,0.20⟩)

5.2. With LDFWG operator

Step 1: Obtain matrix $D=(\aleph_{ij}^{\Bbbk})_{m\times n}$ by DM, which is shown in Table 4.

Table 4
Rating given by DM

	$\breve{\mathscr{G}}^{\zeta}_{1}$	$\breve{\mathscr{G}}^{\zeta}{}_{2}$	$\breve{\mathscr{G}}^{\zeta}{}_{3}$	$\breve{\mathscr{G}}^{\zeta}{}_{4}$
$\mathscr{T}_1^{\lambda}$	(⟨0.50,0.85⟩,⟨0.30,0.10⟩)	(⟨0.45,0.70⟩,⟨0.25,0.20⟩)	(⟨0.65,0.75⟩,⟨0.45,0.25⟩)	(⟨0.85,0.80⟩,⟨0.40,0.20⟩)
$\mathscr{T}_2^{\lambda}$	(⟨0.80,0.90⟩,⟨0.45,0.15⟩)	(⟨0.45,0.65⟩,⟨0.55,0.35⟩)	(⟨0.75,0.45⟩,⟨0.40,0.30⟩)	(⟨0.65,0.85⟩,⟨0.45,0.35⟩)
$\mathscr{T}_3^{\lambda}$	(⟨0.35,0.65⟩,⟨0.50,0.20⟩)	(⟨0.65,0.95⟩,⟨0.25,0.65⟩)	(⟨0.45,0.90⟩,⟨0.30,0.45⟩)	(⟨0.55,0.95⟩,⟨0.50,0.30⟩)
$\mathscr{T}_4^{\lambda}$	(⟨0.50,0.50⟩,⟨0.50,0.25⟩)	(⟨0.90,0.55⟩,⟨0.50,0.40⟩)	(⟨0.45,0.65⟩,⟨0.35,0.50⟩)	(⟨0.35,0.65⟩,⟨0.30,0.20⟩)

Table 5
Normalized LDF decision matrix

	$\breve{\mathscr{G}}^{\zeta}_{1}$	$\breve{\mathscr{G}}^{\zeta}{}_{2}$	$\breve{\mathscr{G}}^{\zeta}{}_{3}$	$\breve{\mathscr{G}}^{\zeta}{}_{4}$
$\mathscr{T}_1^{\lambda}$	(⟨0.50,0.85⟩,⟨0.30,0.10⟩)	(⟨0.70,0.45⟩,⟨0.20,0.25⟩)	(⟨0.65,0.75⟩,⟨0.45,0.25⟩)	(⟨0.85,0.80⟩,⟨0.40,0.20⟩)
$\mathscr{T}_2^{\lambda}$	(⟨0.80,0.90⟩,⟨0.45,0.15⟩)	(⟨0.65,0.45⟩,⟨0.35,0.55⟩)	(⟨0.75,0.45⟩,⟨0.40,0.30⟩)	(⟨0.65,0.85⟩,⟨0.45,0.35⟩)
$\mathscr{T}_3^{\lambda}$	(⟨0.35,0.65⟩,⟨0.50,0.20⟩)	(⟨0.95,0.65⟩,⟨0.65,0.25⟩)	(⟨0.45,0.90⟩,⟨0.30,0.45⟩)	(⟨0.55,0.95⟩,⟨0.50,0.30⟩)
$\mathscr{T}_4^{\lambda}$	(⟨0.50,0.50⟩,⟨0.50,0.25⟩)	(⟨0.55,0.90⟩,⟨0.40,0.50⟩)	(⟨0.45,0.65⟩,⟨0.35,0.50⟩)	(⟨0.35,0.65⟩,⟨0.30,0.20⟩)

Step 2: In this case, $\breve{\mathscr{G}}^{\zeta}{}_{2}$ criteria is cost type criteria all are the benefits types, so there is need of normalization. Normalized LDF decision matrix given in Table 5.

Step 3: Aggregate the LDF values $\mathscr{R}^{S}{}_{ij}$ for all $\mathscr{T}_i^{\lambda}$ using LDFWG operator, given in Table 6.

Table 6
LDF-aggregated values $\mathscr{R}^{S}{}_{i}$

$\mathscr{R}^{S}{}_{1}$	(⟨0.547045,0.771117⟩,⟨0.315797,0.18666⟩)
$\mathscr{R}^{S}{}_{2}$	(⟨0.581468,0.547835⟩,⟨0.469927,0.700454⟩)
$\mathscr{R}^{S}{}_{3}$	(⟨0.442722,0.812834⟩,⟨0.547528,0.796085⟩)
$\mathscr{R}^{S}{}_{4}$	(⟨0.461491,0.670541⟩,⟨0.503649,0.468701⟩)

Step 4: Compute the score for all LDF-aggregated values $\mathscr{R}^{S}{}_{i}$.

$$\breve{\beth}^{\lambda}(\mathscr{R}^{S}{}_{1}) = 0.476266$$

$$\breve{\beth}^{\lambda}(\mathscr{R}^{S}{}_{2}) = 0.480777$$

$$\breve{\beth}^{\lambda}(\mathscr{R}^{S}{}_{3}) = 0.345333$$

$$\breve{\beth}^{\lambda}(\mathscr{R}^{S}{}_{4}) = 0.456474$$

Step 5: Ranks according to SFs.

$$\mathscr{R}^{S}{}_{2} \succ \mathscr{R}^{S}{}_{1} \succ \mathscr{R}^{S}{}_{4} \succ \mathscr{R}^{S}{}_{3}$$

So,

$$\mathscr{T}_2^{\lambda} \succ \mathscr{T}_1^{\lambda} \succ \mathscr{T}^{\lambda}{}_{4} \succ \mathscr{T}_3^{\lambda}$$

$\mathscr{T}_2^{\lambda}$ is best alternative among all other alternatives.

6. Conclusion

MCDM is a significant real-world decision issue, and its most fundamental and essential research is the expression of imprecise information. IFSs, PFSs, and q-ROFSs are all effective methods for handling fuzzy information. LDFSs are more generic than IFS, PFS, and q-ROFS due to their ability to loosen the severe limitations of IFS, PFS, and q-ROFS by considering RPs. MCDM is a crucial subfield in operation research. This assignment's techniques mostly rely on the nature of the issue being evaluated. Our everyday occurrences include unpredictability, imprecision, and obscurity. Existing structures were constructed on the basis of the concept that DMs consider specific limitations while assessing various choices and qualities. However, this kind of situation makes it difficult for DMs to allocate MSDs and NMSDs; therefore, they do so with different constraints. LDFS is a novel method to uncertainty and decision-making issues that incorporates pairs of RPs versus MSDs and NMSDs in order to loosen these limits. We have used LDFSs to assess the validity of DMs' knowledge in the basic framework and to remove any distortion in the decision analysis. The significant advantage of including RPs into the examination is to reduce the likelihood of theoretical knowledge-based MSD and NMSD-related mistakes. In addition, we have developed a number of AOs, including the LDFWA operator and the LDFWG operator. Numerous intriguing aspects of the suggested operators are investigated, and their illustration is convincingly shown.

Ethical Statement

This study does not contain any studies with human or animal subjects performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

Data Availability Statement

Data available on request from the corresponding author upon reasonable request.

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