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Numerical Analysis of Differential Equation with Type-2 Fuzzy Number as Initial Condition

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Abstract: The primary intention of this article is to study numerical solutions of differential equation with interval type-2 fuzzy number as the initial condition. The differential equation is first redrafted in the parametric form; then, it is restructured into three systems of linear differential equations. Each system includes two concurrent linear differential equations with respective initial conditions. The classical fourth-order Runge–Kutta method is developed for the above-derived systems. The ability of the method is corroborated by illustrating the problems.

Keywords: fuzzy differential equation, type-2 fuzzy set, numerical analysis, Runge–Kutta method

1. Introduction

Many of the real-world problems are imprecise in nature. Fuzzy set theory (Zadeh, 1965) was introduced to overcome the impreciseness of the problems/data. This theory assigned a value between "0" and "1" including them to each and every element of the universal set with respect to the subset taken for the study. These values indicate the degrees of the memberships of the elements to the subset. However, these degrees of memberships are again crisp numbers. Hence, in many cases, they do not serve to remove/consider the impreciseness of the problems/data. Consequently, a new fuzzy set was introduced in Zadeh (1975), known as type-2 fuzzy set (T2FS), in which the membership function (MF) is another fuzzy set.

A distinct case of T2FS is an interval-valued T2FS (IVT2FS). IVT2FSs have been applied in several fields like E-commerce (Wu & Liu, 2020), big-data analysis (Shukla et al., 2020), military information (Hanratty et al., 2019), evaluation of Internet of Things-centered healthcare device (Cagri Tolga, 2020), to diagnose fault in gas turbines (Morteza et al., 2020), and multiple criteria decision-making problems (Mohamadghasemi et al., 2020; Chen et al., 2013; Deveci et al., 2020). Some more applications can be found in Jagiello et al. (2022), Andul-Sadah et al. (2022), Starczewski et al. (2022), Lv et al. (2021), Qin et al. (2022), and Zakaria et al. (2021). A complete literature review on T2FS and its real-time applications can be found in De et al. (2022).

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Pioneering application of the theory of fuzzy sets can be found in the field of differential equations. The advancement of differential equation with type-1 fuzzy environment and a variety of methods, particularly numerical methods to simulate the solution of fuzzy differential equation (FDE), can be studied in the literature. The progression of finding solutions of FDE is still an ongoing progress in the field of research. However, a very few have attempted to study differential equation in type-2 fuzzy background.

The method to differentiate type-2 fuzzy logic systems was introduced and premeditated in Mendel (2004). The derivatives of type-2 fuzzy number (T2FN)-valued functions have been studied in Mazandarani and Najariyan (2014) and used it to analyzes the problems, raised in the field of Electrical Engineering, Environment Engineering, and Economics.

The study of first-order FDE on the space of quasi-T2FSs (QT2FS) can be found in (Kardan et al., 2016). Pulp and paper industry problem and electrical engineering problem under type-2 fuzzy environment have been taken for study in Najariyan et al. (2017).

Bandyopadhyay and Kar (2018) have modeled natural problems such as prey–predator model encompassing different circumstances and Lorenz model with type-2 fuzzy initial conditions. The authors converted the type-2 FDE (T2FDE) of these models into system of linear differential equations and simulated the solutions. In Debnath et al. (2018), FDE methodology has been implemented to study inventory problem in which the demand has been taken as a T2FN. In this article, a T2FDE in parametric form is taken. It is altered into three systems of ordinary differential equations: (1) related to lower MF (LMF), (2) related to upper MF (UMF), and (3) related to principal set

(PS). Each system consists of two simultaneous ordinary differential equations. Numerical solutions of these equations are obtained by classical Runge–Kutta method of order 4.

Section 2 provides necessary basic concepts on T2FSs and T2FNs and its metric. T2FDEs are considered in Section 3. Section 4 develops the classical fourth-order Runge–Kutta method to obtain numerical solution of T2FDE. Numerical problems have been analyzed in Section 5, and the deduction is given in Section 6.

2. Basic Concepts

Definition 2.1. (Liang & Mendel, 2000). A T2FS is defined as $\widetilde{W} = \{((a,u), \mu_{\widetilde{W}}(a,u))/\ a \in X,\ \ \forall u \in J_a\ \subset [0,1]\ \}$, where $0 \le \mu_{\widetilde{W}}(a,u) \le 1$. The variable "$a \in X$" is called primary variable of $\widetilde{W}$ and "$u \in J_a$" is called the secondary variable of $\widetilde{W}$. At each fixed point $a \in X$, J_a is termed as primary membership of "a" and "$\mu_{\widetilde{W}}(a,u) = \mu_{\widetilde{W}}(a)$" called its secondary MF.

Definition 2.2. (Mendel et al., 2006). The footprint of uncertainty (FOU) of a T2FS "$\widetilde{W}$" is a bounded region which encloses the vagueness in the primary membership of "$\widetilde{W}$".

Definition 2.3. (Mendel et al., 2006). The bounds of the FOU of a T2FS "$\widetilde{W}$" are two type-1 MFs and are called a UMF and a LMF, respectively. The UMF is related to the upper bound of $FOU(\widetilde{W})$, and it is symbolized by "$\bar{\mu}_{\widetilde{W}}(a)$". The LMF is connected with the lower bound of $FOU(\widetilde{W})$, and it is symbolized by "$\underline{\mu}_{\widetilde{W}}(a)$".

(i.e.) $\bar{\mu}_{\widetilde{W}}(a) = \overline{FOU(\widetilde{W})}$,$\forall a \in X$ and $\underline{\mu}_{\widetilde{W}}(a) = \underline{FOU(\widetilde{W})}$,$\forall a \in X$.

Using the concept of UMF and LMF, J_a can be expressed as: $J_a = \{(a,u):\ \forall u \in [LMF_{FOU(\widetilde{W})}(a),\ UMF_{FOU(\widetilde{W})}(a)] \subset [0,1]\}$.

Definition 2.4. (Mendel, 2001). An IVT2FS is defined as $\widetilde{W} = \{((a,u), \mu_{\widetilde{W}}(a,u))/\ a \in X,\ \ \forall u \in J_a\ \subset [0,1]\ \}$, where $\mu_{\widetilde{W}}(a,u) = 1$.

Definition 2.5. (Zhai & Mendel, 2011). Let "$\widetilde{W}$" be a T2FS. For a fixed point "$a_0 \in X$", $\mu_{\widetilde{W}}(a_0) = \mu_{\widetilde{W}}(a_0, u)$ is termed as the secondary MF or vertical slice of $\widetilde{W}$ at "a_0" and is expressed as: $\mu_{\widetilde{W}}(a_0) = \int_{u \in J_{a_0}} f_{a_0}(u)/u$, where "$f_{a_0}(u)$" is called secondary membership grade and "$\mu_{\widetilde{W}}(a_0)$" is a type-1 fuzzy set.

Definition 2.6. (Hamrawi, 2016). An α-plane for a T2FS, $\widetilde{W}$, is defined as:
$\widetilde{W}_\alpha = \{(a,u), \mu_{\widetilde{W}}(a,u)\alpha\ /\ a \in X,\ \ \forall u \in J_a\ \subset [0,1]$.

If "$S_{\widetilde{W}}(a \mid \alpha)$" denotes an α-cut of the secondary MF$\mu_{\widetilde{W}}(a)$, then $S_{\widetilde{W}}(a|\ \alpha) = [s_L(a\ |\alpha), s_R(a\ |\alpha)]$.

Definition 2.7. (Hamrawi, 2016). Let "$\widetilde{W}$" be a T2FS. Let $\widetilde{W}_\beta = \left(\underline{\widetilde{W}_\beta}, \overline{\widetilde{W}_\beta}\right)$ be the β-plane of $\widetilde{W}$. Then, $\alpha-$ cut representation of "$\widetilde{W}_\beta$" is given by $\widetilde{W}_\beta{}^\alpha = \left(\underline{\widetilde{W}_\beta{}^\alpha}, \overline{\widetilde{W}_\beta{}^\alpha}\right)$, where $\underline{\widetilde{W}_\beta{}^\alpha} = LMF_{FOU(\widetilde{W})}(a;\alpha,\beta)$ and

$\overline{\widetilde{W}_\beta{}^\alpha} = UMF_{FOU(\widetilde{W})}(a;\alpha,\beta),\ \forall a \in X.$

Note 2.1 $FOU(\widetilde{W}) = \cup_{a \in X} J_a = \widetilde{W}_0$.

Definition 2.8. (Hamrawi, 2016). The set of primary grades whose secondary grades equal to "1" is called as the PS of "$\widetilde{W}$" and is given by: $PS(\widetilde{W}) = \{(a,u)/\ a \in X, u \in J_a, f_a(u) = 1\} = \widetilde{W}_1$.

Definition 2.9. (Hamrawi, 2016). A T2FS "$\widetilde{W}$" defined on "R", the set of real numbers, is called as a perfect T2FN (PT2FN) if both UMF and LMF of FOU$(\widetilde{W})$ are type-1 fuzzy numbers (T1FNs).

Definition 2.10. (Hamrawi, 2016). A QT2FS is a T2FS which is entirely determined by its FOU and PS.

Definition 2.11. (Hamrawi, 2016). A perfect quasi-T2FN (PQT2FN) is a PT2FN which is totally determined by its FOU and PS.

Definition 2.12. (Hamrawi, 2016). Let $\tilde{A}$ and $\tilde{B}$ be two T2FS. Then, $\tilde{A} = \tilde{B}$ iff $S_{\tilde{A}}(a \mid \alpha) = S_{\tilde{B}}(a \mid \alpha)$, $\forall a \in X,\ \ \alpha \in [0,1]$.

Definition 2.13. (Hung & Yang, 2004). Let $\tilde{A}$ and $\tilde{B}$ be two T2FS. Let $x_0 \in [a,b] \subseteq X$ and let $S_{\tilde{A}}(x_0\ |\alpha)$ and $S_{\tilde{B}}(x_0 \mid \alpha)$ represent the α-cuts of the secondary MFs $\mu_{\tilde{A}}(x_0)$ and $\mu_{\tilde{B}}(x_0)$, respectively. Then, the distance between $\tilde{A}$ and $\tilde{B}$ is defined as follows: $d_2(\tilde{A},\tilde{B}) = \int_a^b H_f(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))dx$, where

$$H_f(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \frac{\int_0^1 \alpha\ d_H(S_{\tilde{A}}(x \mid \alpha), S_{\tilde{B}}(x \mid \alpha))d\alpha}{\int_0^1 \alpha\ d\alpha} = 2\int_0^1 \alpha\ d_H(S_{\tilde{A}}(x \mid \alpha), S_{\tilde{B}}(x \mid \alpha))d\alpha$$

Preposition 2.1. (Hung & Yang, 2004). The distance measure d_2 defined in Definition 2.13 is a metric on the space of T2FSs.

Proof: Refer Hung and Yang (2004).

Preposition 2.2. (Figueroa et al., 2015). Let $\tilde{A}$, $\tilde{B}$ be two interval-valued T2FNs. Let $\tilde{A}_\beta = \left(\underline{\tilde{A}_\beta}, \overline{\tilde{A}_\beta}\right)$ be the β-plane of $\tilde{A}$ and let $\tilde{B}_\beta = \left(\underline{\tilde{B}_\beta}, \overline{\tilde{B}_\beta}\right)$ be the β-plane of $\tilde{B}$. Then, at $\alpha-$ level, "$\tilde{A}_\beta$" is represented by

$\tilde{A}_\beta{}^\alpha = (LMF_{FOU(\tilde{A})}(x;\alpha,\beta), UMF_{FOU(\tilde{A})}(x;\alpha,\beta)),\quad \forall x \in R.$ (i.e.) $\tilde{A}_\beta{}^\alpha = \left(\left[\tilde{A}^\alpha{}_{L(-)}, \tilde{A}^\alpha{}_{R(-)}\right], \left[\tilde{A}^\alpha{}_{L(+)}, \tilde{A}^\alpha{}_{R(+)}\right]\right)$.

Similarly, at $\alpha-$ level, "$\tilde{B}_\beta$" is represented by $\tilde{B}_\beta{}^\alpha = \left(\left[\tilde{B}^\alpha{}_{L(-)},\ \tilde{B}^\alpha{}_{R(-)}\right], \left[\tilde{B}^\alpha{}_{L(+)}, \tilde{B}^\alpha{}_{R(+)}\right]\right)$.

The distance (metric) d_α between $\tilde{A}$ and $\tilde{B}$ is given on a set of "n" α-cuts, $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ and $\Lambda = \sum_{i=1}^n \alpha_i$, is $d_\alpha(\tilde{A},\tilde{B}) \triangleq \frac{1}{\Lambda}\sum_{i=1}^n \alpha_i\left\{\left|\tilde{A}^{\alpha_i}{}_{L(+)} - \tilde{B}^{\alpha_i}{}_{L(+)}\right| + \left|\tilde{A}^{\alpha_i}{}_{L(-)} - \tilde{B}^{\alpha_i}{}_{L(-)}\right| + \left|\tilde{A}^{\alpha_i}{}_{R(+)} - \tilde{B}^{\alpha_i}{}_{R(+)}\right| + \left|\tilde{A}^{\alpha_i}{}_{R(-)} - \tilde{B}^{\alpha_i}{}_{R(-)}\right|\right\}$.

If α is continuous, then $\Lambda = \int_0^1 \alpha d\alpha$, so d_α is defined as $d_\alpha(\tilde{A},\tilde{B}) \triangleq 2\int_0^1 \alpha \left\{\begin{array}{c} \left|\tilde{A}^\alpha{}_{L(+)} - \tilde{B}^\alpha{}_{L(+)}\right| + \left|\tilde{A}^\alpha{}_{L(-)} - \tilde{B}^\alpha{}_{L(-)}\right| \\ +\left|\tilde{A}^\alpha{}_{R(+)} - \tilde{B}^\alpha{}_{R(+)}\right| + \left|\tilde{A}^\alpha{}_{R(-)} - \tilde{B}^\alpha{}_{R(-)}\right| \end{array}\right\} d\alpha.$

Proof: (Figueroa et al., 2015).

Let E_2 be the collection of all triangular PQT2FNs. Suppose that $\hat{y} \in E_2$ is a triangular PQT2FN with core "y".

(i.e.) $\widetilde{y} = [y_{lU}, y_l, y_{lL}, y, y_{rL}, y_r, y_{rU}] \in E_2$ is a triangular PQT2FN, where, y_{lL} ,y_{rL} are the two end points of the support of $LMF_{FOU(\widetilde{y})}$; y_{lU} ,y_{rU} are the left and right end points of the support of $UMF_{FOU(\widetilde{y})}$; and y_l ,y_r are the left and right end points of the support of $PS(\widetilde{y})$.

(i.e.) $LMF_{FOU(\widetilde{y})} = [y_{lL}, y, y_{rL}]$; $UMF_{FOU(\widetilde{y})} = [y_{lU}, y, y_{rU}]$; and $PS(\widetilde{y}) = [y_l, y, y_r]$ and all these are T1FNs.

The $\alpha-$ cut of $LMF_{FOU(\widetilde{y}(t))}$, $UMF_{FOU(\widetilde{y}(t))}$, and $PS(\widetilde{y}(t))$ is respectively given as follows:

$$[y_{lL}, y, y_{rL}]^\alpha = \begin{bmatrix} y-(1-\alpha)(y-y_{lL}), \\ y+(1-\alpha)(y_{rL}-y) \end{bmatrix}; [y_{lU}, y, y_{rU}]^\alpha = \begin{bmatrix} y-(1-\alpha)(y-y_{lU}), \\ y+(1-\alpha)(y_{rU}-y) \end{bmatrix}; \text{and}$$

$$[y_l, y, y_r]^\alpha = [y-(1-\alpha)(y-y_l), y+(1-\alpha)(y_r-y)].$$

The β-plane of "$\widetilde{y}$" at $\alpha-$ level is given by $\widetilde{y}_\beta{}^\alpha = \left(\underline{\widetilde{y}_\beta}{}^\alpha, \overline{\widetilde{y}_\beta}{}^\alpha\right)$, where

$$\begin{aligned}\underline{\widetilde{y}_\beta}{}^\alpha &= [y_{lL}, y, y_{rL}]^\alpha \\ &= \begin{bmatrix} y-(1-\alpha)(y-y_l)-(1-\beta)(1-\alpha)(y_l-y_{lL}), \\ y+(1-\alpha)(y_r-y)+(1-\beta)(1-\alpha)(y_r-y_{rL}) \end{bmatrix}; \text{and}\end{aligned}$$

$$\begin{aligned}\overline{\widetilde{y}_\beta}{}^\alpha &= [y_{lU}, y, y_{rU}]^\alpha{}_\beta \\ &= \begin{bmatrix} y-(1-\alpha)(y-y_l)-(1-\beta)(1-\alpha)(y_l-y_{lU}), \\ y+(1-\alpha)(y_r-y)+(1-\beta)(1-\alpha)(y_{rU}-y_r) \end{bmatrix}.\end{aligned}$$

3. Differential of Type-2 Fuzzy Functions

Following the description of generalized differentiability of fuzzy-valued functions, Bede & Gal (2005 and Mazandarani & Najariyan (2014) have defined the derivative of T2FN-valued functions.

Definition 3.1. (Mazandarani & Najariyan, 2014). Let $\tilde{e}_1, \tilde{e}_2 \in E_2$. If there exist $\tilde{e}_3 \in E_2$ such that $\tilde{e}_1 = \tilde{e}_2 + \tilde{e}_3$, then $\tilde{e}_3$ is named as type-2 H-difference of $\tilde{e}_1$ and $\tilde{e}_2$, represented by $\tilde{e}_1 \; \underline{H_2} \; \tilde{e}_2$.

Theorem 3.1. (Mazandarani & Najariyan, 2014). If $\tilde{e}_1, \tilde{e}_2 \in E_2$, then $\beta-$ plane of the H_2-difference of $\tilde{e}_1$ and $\tilde{e}_2$ is the H-difference of LMF and UMF of $\tilde{e}_1$ and $\tilde{e}_2$.

Proof: Refer Mazandarani and Najariyan (2014).

Definition 3.2. (Mazandarani & Najariyan, 2014). Let $T = [a, b] \subseteq R$. Then $\tilde{F} : T \to E_2$ is called a triangular PQT2FN-valued function.

Definition 3.3. (Mazandarani & Najariyan, 2014). Let $\tilde{F} : T \subset R \to E_2$ and let $x_0 \in T$. Then, $\tilde{F}$ is said to be differentiable at x_0 if there exists an $\tilde{F}'(x_0) \in E_2$ such that, for all $\delta > 0$ sufficiently near to 0, then

(1) There are $\tilde{F}(x_0+\delta) \; \underline{H_2} \; \tilde{F}(x_0)$, $\tilde{F}(x_0) \; \underline{H_2} \tilde{F}(x_0-\delta)$, as well as the limits:

$$lim_{\delta\to 0}\frac{\tilde{F}(x_0+\delta)\; \underline{H_2}\tilde{F}(x_0)}{\delta} = lim_{\delta\to 0}\frac{\tilde{F}(x_0)\; \underline{H_2}\; \tilde{F}(x_0-\delta)}{\delta} = \tilde{F}'(x_0) \text{ (or)}$$

(2) There are $\tilde{F}(x_0) \; \underline{H_2} \; \tilde{F}(x_0+\delta)$, $\tilde{F}(x_0-\delta) \; \underline{H_2} \; \tilde{F}(x_0)$, as well as the limits:

$$lim_{\delta\to 0}\frac{\tilde{F}(x_0)\; \underline{H_2}\tilde{F}(x_0+\delta)}{-\delta} = lim_{\delta\to 0}\frac{\tilde{F}(x_0-\delta)\; \underline{H_2}\tilde{F}(x_0)}{-\delta} = \tilde{F}'(x_0) \text{ (or)}$$

(3) There are $\tilde{F}(x_0+\delta) \; \underline{H_2}\tilde{F}(x_0)$, $\tilde{F}(x_0-\delta) \; \underline{H_2}\tilde{F}(x_0)$, as well as the limits:

$$lim_{\delta\to 0}\frac{\tilde{F}(x_0+\delta)\; \underline{H_2}\; \tilde{F}(x_0)}{\delta} = lim_{\delta\to 0}\frac{\tilde{F}(x_0-\delta)\; \underline{H_2}\; \tilde{F}(x_0)}{-\delta} = \tilde{F}'(x_0) \text{ (or)}$$

(4) There are $\tilde{F}(x_0) \; \underline{H_2} \; \tilde{F}(x_0+\delta)$, $\tilde{F}(x_0) \; \underline{H_2} \; \tilde{F}(x_0-\delta)$, as well as the limits:

$$lim_{\delta\to 0}\frac{\tilde{F}(x_0)\; \underline{H_2}\; \tilde{F}(x_0+\delta)}{-\delta} = lim_{\delta\to 0}\frac{\tilde{F}(x_0)\; \underline{H_2}\; \tilde{F}(x_0-\delta)}{\delta} = \tilde{F}'(x_0).$$

Note 3.1. In Definition 3.3, the limits have been taken in the metric space (E_2, d_2).

Note 3.2. $\tilde{F}$ (t) is differentiable in the first form," if it is differentiable as in Definition 3.3(1), and differentiable in the second form," if it is differentiable as in Definition 3.3(2).

Theorem 3.2. (Mazandarani & Najariyan, 2014). Suppose that $\widetilde{G} : (a, b) \to E_2$ and let $\left[\widetilde{G}(t)\right]_\beta = \left[\underline{\widetilde{G}_\beta}(t), \overline{\widetilde{G}_\beta}(t)\right]$, for each $0 \le \beta \le 1$, where $\underline{\widetilde{G}_\beta}(t)$, are LMF and UMF of $\beta-$ plane of $\widetilde{G}(t)$, respectively, then

1. If $\widetilde{G}$ is H_2- differentiable in the first form and $\underline{\widetilde{G}_\beta}(t), \overline{\widetilde{G}_\beta}(t)$ are differential functions in the first form Bede & Gal (2005), then $\left[\widetilde{G}'(t)\right]_\beta = \left[\underline{\widetilde{G}'_\beta}(t), \overline{\widetilde{G}'_\beta}(t)\right]$.

2. If $\widetilde{G}$ is H_2- differentiable in the second form and $\underline{G_\beta}(t), \overline{\widetilde{G}_\beta}(t)$ are differential functions in the second form Bede & Gal (2005), then $\left[\widetilde{G}'(t)\right]_\beta = \left[\underline{\widetilde{G}'_\beta}(t), \overline{\widetilde{G}'_\beta}(t)\right]$.

Proof: Refer Mazandarani and Najariyan (2014).

4. Fourth-Order Runge–Kutta Method for Type-2 Fuzzy Initial Value Problems

Consider a type-2 fuzzy initial value problem (T2FIVP)

$$\widetilde{y}'(t) = \tilde{g}(t, \widetilde{y}(t)), t \in [c, d], \widetilde{y}(c) = \widetilde{y}_0. \tag{1}$$

The problem is to find the numerical answers to Equation (1) by Runge–Kutta method of order 4.

4.1. Differentiable in the first form

Assume that the T2FIVP given in Equation (1) is H_2- differentiable in the first form.

Let the analytical solution of Equation (1) in the parametric form be

$[\widetilde{y}(t)]^\alpha_\beta = [Y_{lU}(t), Y_l(t), Y_{lL}(t), Y(t), Y_{rL}(t), Y_r(t), \; Y_{rU}(t)]^\alpha_\beta$ and the approximate solution to Equation (1) by Runge–Kutta method of order 4 be $[\widetilde{y}(t)]^\alpha_\beta = [y_{lU}(t), y_l(t), y_{lL}(t), \; y(t), y_{rL}(t), y_r(t), y_{rU}(t)]^\alpha_\beta$.

Here, $[Y_{lU}(t), Y_{rU}(t)]^\alpha_\beta$; $[Y_l(t), Y_r(t)]^\alpha_\beta$; $[Y_{lL}(t), Y_{rL}(t)]^\alpha_\beta$; $[y_{lU}(t), y_{rU}(t)]^\alpha_\beta$; $[y_l(t), y_r(t)]^\alpha_\beta$ and $[y_{lL}(t), y_{rL}(t)]^\alpha_\beta$ are valid T1FNs.

To approximate $[y_{lU}(t), y_{rU}(t)]^\alpha_\beta$:
Let $a = t_0 \le t_1 \le t_2 \le \ldots\ldots\ldots\ldots\ldots\ldots \le t_N = b$ and $h = \frac{(b-a)}{N} = t_{n+1} - t_n$.

The analytical and estimated solutions at $t_n, 0 \le n \le N$ are denoted by $[Y_{lU}(t_n), Y_{rU}(t_n)]^\alpha_\beta$ and $[y_{lU}(t_n), y_{rU}(t_n)]^\alpha_\beta$, respectively. By Runge–Kutta method of order 4,

$$y_{lU}(t_{n+1};\alpha,\beta) = y_{lU}(t_n;\alpha,\beta) + \frac{1}{6}\left\{\begin{array}{l} \underline{k_1}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) + 2\underline{k_2}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) \\ +2\underline{k_3}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) + \underline{k_4}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) \end{array}\right\} \tag{2}$$

$$y_{rU}(t_{n+1};\alpha,\beta) = y_{rU}(t_n;\alpha,\beta) + \frac{1}{6}\left\{\begin{array}{l} \overline{k_1}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) + 2\overline{k_2}\left(t_n, [y_{lU}(t_n), y_r(t_n)]_\beta^\alpha\right) \\ +2\overline{k_3}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) + \overline{k_4}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) \end{array}\right\} \tag{3}$$

where

$$[k_i(t_n, [y_{lU}(t_n), y_{rU}(t_n)])]_\beta^\alpha = \Big[\underline{k_i}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]), \overline{k_i}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)])\Big],\ i = 1,2,3,4. \tag{4}$$

$$\underline{k_1}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) = min\{h.\tilde{f}(t_n, u)\ \backslash u \in [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]\} \tag{5}$$

$$\overline{k_1}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) = max\{h.\tilde{f}(t_n, u)\ \backslash u \in [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]\} \tag{6}$$

$$\underline{k_2}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) = min\left\{ h.\tilde{f}\left(t_n + \tfrac{h}{2}, u + \tfrac{v}{2}\right) \backslash v \in [k_1(t_n, [y_{lU}(t_n), y_{rU}(t_n)])]_\beta^\alpha \right\} \tag{7}$$

$$\overline{k_2}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) = max\left\{ h.\tilde{f}\left(t_n + \frac{h}{2}, u + \frac{v}{2}\right) \backslash\ v \in [k_1(t_n, [y_{lU}(t_n), y_{rU}(t_n)])]_\beta^\alpha \right\} \tag{8}$$

$$\underline{k_3}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) = min\left\{ h.\tilde{f}\left(t_n + \frac{h}{2}, u + \frac{v}{2}\right) \backslash\ v \in [k_2(t_n, [y_{lU}(t_n), y_{rU}(t_n)])]_\beta^\alpha \right\} \tag{9}$$

$$\overline{k_3}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) = max\left\{ h.\tilde{f}\left(t_n + \frac{h}{2}, u + \frac{v}{2}\right) \backslash\ v \in [k_2(t_n, [y_{lU}(t_n), y_{rU}(t_n)])]_\beta^\alpha \right\} \tag{10}$$

$$\underline{k_4}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) = min\left\{ h.\tilde{f}(t_n + h, u + v) \backslash\ v \in [k_3(t_n, [y_{lU}(t_n), y_{rU}(t_n)])]_\beta^\alpha \right\} \tag{11}$$

$$\overline{k_4}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) = max\left\{ h.\tilde{f}(t_n + h, u + v) \backslash\ v \in [{}_3(t_n, [y_{lU}(t_n), y_{rU}(t_n)])]_\beta^\alpha \right\} \tag{12}$$

Define,

$$F\Big[t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\Big] = \frac{1}{6}\left\{\begin{array}{l} \underline{k_1}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) + 2\underline{k_2}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) \\ +2\underline{k_3}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) + \underline{k_4}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) \end{array}\right\} \tag{13}$$

$$G\Big[t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\Big] = \frac{1}{6}\left\{\begin{array}{l} \overline{k_1}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) + 2\overline{k_2}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) \\ +2\overline{k_3}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) + \overline{k_4}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]_\beta^\alpha\right) \end{array}\right\} \tag{14}$$

Let

$$Y_{lU}(t_{n+1};\alpha,\beta) = Y_{lU}(t_n;\alpha,\beta) + F[t_n, [Y_{lU}(t_n;\alpha,\beta), Y_{rU}(t_n;\alpha,\beta)]] \tag{15}$$

$$Y_{rU}(t_{n+1};\alpha,\beta) = Y_{rU}(t_n;\alpha,\beta) + G[t_n, [Y_{lU}(t_n;\alpha,\beta), Y_{rU}(t_n;\alpha,\beta)]] \tag{16}$$

And,

$$y_{lU}(t_{n+1};\alpha,\beta) = y_{lU}(t_n;\alpha,\beta) + F[t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]] \tag{17}$$

$$y_{rU}(t_{n+1};\alpha,\beta) = y_{rU}(t_n;\alpha,\beta) + G[t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]] \tag{18}$$

The lemmas and theorems in Ma et al. (1999) can be used to show the convergences of theses approximates.

(i.e.) $Y_{lU}(t;\alpha,\beta) = \lim_{h\to 0} y_{lU}(t;\alpha,\beta)$ and $Y_{rU}(t;\alpha,\beta) = \lim_{h\to 0} y_{rU}(t;\alpha,\beta)$.

Similarly, we can find approximate solutions to $[Y_l(t), Y_r(t)]_\beta^\alpha$; $[Y_{lL}(t), Y_{rL}(t)]_\beta^\alpha$.

4.2. Differentiable in the second form

Assume that $\tilde{y}'(t;\alpha,\beta)$ is H_2- differentiable in the second form. Let the analytical solution of Equation (1) be $[\tilde{y}(t)]_\beta^\alpha = [Y_{lU}(t), Y_l(t), Y_{lL}(t), Y(t), Y_{rL}(t), Y_r(t), Y_{rU}(t)]_\beta^\alpha$ and the approximate solution to Equation (1) by Runge–Kutta method of order 4 be

$[\tilde{y}(t)]_\beta^\alpha = [y_{lU}(t), y_l(t), y_{lL}(t), y(t), y_{rL}(t), y_r(t), y_{rU}(t)]_\beta^\alpha$. Here, $[Y_{lU}(t), Y_{rU}(t)]_\beta^\alpha$; $[Y_l(t), Y_r(t)]_\beta^\alpha$; $[Y_{lL}(t), Y_{rL}(t)]_\beta^\alpha$

$[y_{lU}(t), y_{rU}(t)]_\beta^\alpha$; $[y_l(t), y_r(t)]_\beta^\alpha$ and $[y_{lL}(t), y_{rL}(t)]_\beta^\alpha$ are valid T1FNs.

To approximate $[y_{lU}(t), y_{rU}(t)]^{\alpha}_{\beta}$:

Let $a = t_0 \le t_1 \le t_2 \le \ldots\ldots\ldots\ldots \le t_N = b$ and $h = \frac{(b-a)}{N} = t_{n+1} - t_n$.

The analytical and estimated solutions at $t_n, 0 \le n \le N$ are denoted by $[Y_{lU}(t_n), Y_{rU}(t_n)]^{\alpha}_{\beta}$ and $[y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}$, respectively. By Runge–Kutta method of order 4,

$$y_{lU}(t_{n+1};\alpha,\beta) = y_{lU}(t_n;\alpha,\beta) + \frac{1}{6}\left\{ \begin{array}{l} \overline{k_1}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}\right) + 2\overline{k_2}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}\right) \\ +2\overline{k_3}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}\right) + \overline{k_4}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}\right) \end{array} \right\} \tag{19}$$

$$y_{rU}(t_{n+1};\alpha,\beta) = y_{rU}(t_n;\alpha,\beta) + \frac{1}{6}\left\{ \begin{array}{l} \underline{k_1}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}\right) + 2\underline{k_2}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}\right) \\ +2\underline{k_3}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}\right) + \underline{k_4}\left(t_n, [y_{lU}(t_n), y_{rU}(t_n)]^{\alpha}_{\beta}\right) \end{array} \right\} \tag{20}$$

where

$[k_i(t_n, [y_{lU}(t_n), y_{rU}(t_n)])]^{\alpha}_{\beta} = \begin{bmatrix} \underline{k_i}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]), \\ \overline{k_i}(t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]) \end{bmatrix}$, $i = 1, 2, 3, 4$, are given from Equation (5) to Equation (12). Using Equations (19) and (20) together with Equations (13) and (14), the analytical solution of the T2FIVP given in Equation (1) is

$$Y_{lU}(t_{n+1};\alpha,\beta) = Y_{lU}(t_n;\alpha,\beta) + G[t_n, [Y_{lU}(t_n;\alpha,\beta), Y_{rU}(t_n;\alpha,\beta)]] \tag{21}$$

$$Y_{rU}(t_{n+1};\alpha,\beta) = Y_{rU}(t_n;\alpha,\beta) + F[t_n, [Y_{lU}(t_n;\alpha,\beta), Y_{rU}(t_n;\alpha,\beta)]] \tag{22}$$

And the approximate solution is given by

$$y_{lU}(t_{n+1};\alpha,\beta) = y_{lU}(t_n;\alpha,\beta) + G[t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]] \tag{23}$$

$$y_{rU}(t_{n+1};\alpha,\beta) = y_{rU}(t_n;\alpha,\beta) + F[t_n, [y_{lU}(t_n;\alpha,\beta), y_{rU}(t_n;\alpha,\beta)]] \tag{24}$$

The lemmas and theorems in Ma et al. (1999) can be used to prove the convergences of theses approximates.

(i.e.) $Y_{lU}(t;\alpha,\beta) = \lim_{h\to 0} y_{lU}(t;\alpha,\beta)$ and $Y_{rU}(t;\alpha,\beta) = \lim_{h\to 0} y_{rU}(t;\alpha,\beta)$.

Similarly, we can find approximate solutions to $[Y_l(t), Y_r(t)]^{\alpha}_{\beta}$; $[Y_{lL}(t), Y_{rL}(t)]^{\alpha}_{\beta}$.

5. Numerical Example

Example 5.1. Consider a T2FIV problem $\tilde{y}'(t) = -\tilde{y}(t)$; $t \in [0, 1]$, with the initial condition

$$\tilde{y}(t_0) = \tilde{y}_0 = [y_{0lU}, y_{0l}, y_{0lL}, y_0, y_{0rL}, y_{0r}, y_{0rU}] = [0.85, 0.9, 0.96, 1, 1.04, 1.1, 1.15] \tag{25}$$

(i.e.) $y_{0lU} = 0.85$; $y_{0l} = 0.9$; $y_{0lL} = 0.96$; $y_0 = 1$; $y_{0rL} = 1.04$; $y_{0r} = 1.1$; $y_{0rU} = 1.15$.

The parametric forms of the initial conditions are given by:

$y_{0lU} = 0.85 + 0.15\alpha + 0.05\beta - 0.05\alpha\beta$;

$y_{0l} = 0.9 + 0.1\alpha$;

$y_{0lL} = 0.94 + 0.06\alpha - 0.04\beta + 0.04\alpha\beta$;

$y_{0rL} = 1.04 - 0.04\alpha + 0.06\beta - 0.06\alpha\beta$;

$y_{0r} = 1.1 - 0.1\alpha$; $y_{0rU} = 1.15 - 0.15\alpha - 0.05\beta + 0.05\alpha\beta$.

Now, let $\tilde{y}(t) = [y_{lU}, y_l, y_{lL}, y, y_{rL}, y_r, y_{rU}]$ and hence $\tilde{y}'(t) = [y'_{lU}, y'_l, y'_{lL}, y', y'_{rL}, y'_r, y'_{rU}]$

$\therefore$ Equation (25) $\Rightarrow$

$$[y'_{lU}, y'_l, y'_{lL}, y', y'_{rL}, y'_r, y'_{rU}] = -[y_{lU}, y_l, y_{lL}, y, y_{rL}, y_r, y_{rU}] \tag{26}$$

Case (1): Presume that $\tilde{y}(t)$ is H_2- differentiable in the first form.

$\therefore$ Equation (26) $\Rightarrow$ $[y'_{lU}, y'_l, y'_{lL}, y', y'_{rL}, y'_r, y'_{rU}] = [-y_{rU}, -y_r, -y_{rL}, -y, -y_{lL}, -y_l, -y_{lU}]$;

$$\Rightarrow y'_{lU} = -y_{rU} \tag{27}$$

$$y'_l = -y_r \tag{28}$$

$$y'_{lL} = -y_{rL} \tag{29}$$

$$y' = -y \tag{30}$$

$$y'_{rL} = -y_{lL} \tag{31}$$

$$y'_r = -y_l \tag{32}$$

$$y'_{rU} = -y_{lU} \tag{33}$$

Pairing up Equations (27) and (33), we get a system of equations

$$\left. \begin{array}{l} y'_{lU} = -y_{rU} \\ y'_{rU} = -y_{lU} \end{array} \right\} \tag{34}$$

with initial conditions: $y_{0lU} = 0.85 + 0.15\alpha + 0.05\beta - 0.05\alpha\beta$; $y_{0rU} = 1.15 - 0.15\alpha - 0.05\beta + 0.05\alpha\beta$.

Pairing up Equations (28) and (32), we get a system of equations

$$\left. \begin{array}{l} y'_l = -y_r \\ y'_r = -y_l \end{array} \right\} \tag{35}$$

with initial conditions: $y_{0l} = 0.9 + 0.1\alpha$; $y_{0r} = 1.1 - 0.1\alpha$.

Pairing up Equations (29) and (31), we get a system of equations

$$\left. \begin{array}{l} y'_{lL} = -y_{rL} \\ y'_{rL} = -y_{lL} \end{array} \right\} \tag{36}$$

with initial conditions: $y_0 = 0.94 + 0.06\alpha - 0.04\beta + 0.04\alpha\beta$; $y_{0rL} = 1.04 - 0.04\alpha + 0.06\beta - 0.06\alpha\beta$.

The analytical solution of UMF (i.e.), the system of equations given in Equation (34), is

$$Y_{\text{lU}} = (-0.15 + 0.15\alpha + 0.05\beta - 0.05\alpha\beta)e^t + e^{-t};$$

$$Y_{\text{rU}} = (0.15 - 0.15\alpha - 0.05\beta + 0.05\alpha\beta)e^t + e^{-t}.$$

The analytical solution of the principal MF given by Equation (35) is

$$Y_{\text{l}} = (-0.10 + 0.10\alpha)e^t + e^{-t};$$

$$Y_{\text{r}} = (0.10 - 0.10\alpha)e^t + e^{-t}.$$

The analytical solution of LMF given by Equation (36) is

$$Y_{\text{lL}} = (-0.05 + 0.05\alpha - 0.05\beta + 0.05\alpha\beta)e^t + (1 + 0.01\alpha + 0.01\beta - 0.01\alpha\beta)e^{-t};$$

$$Y_{\text{rL}} = (0.05 - 0.05\alpha + 0.05\beta - 0.05\alpha\beta)e^t + (1 + 0.01\alpha + 0.01\beta - 0.01\alpha\beta)e^{-t}.$$

It can be seen that for $\beta = 1$, the UMF, the principal membership, and LMF are same.

By fixing $''\beta = 0''$ and letting $''\alpha''$ to vary in [0, 1], the approximate solutions of UMF (AUMD11) and LMF (ALMD11) are obtained by the method that has been proposed in Section 4 (Runge–Kutta method of order 4) for $''t = 1''$ with $h = 0.1$.

Similarly, the approximate solutions of UMF (AUMD12, AUMD13) and LMF (ALMD12, ALMD13) are simulated for $\beta = 0.5$ and $\beta = 1$. All these approximations are plotted in Figure 1.

The errors (ED11, ED12, and ED13) between the approximate solutions and analytical solutions are plotted in Figure 2.

Clearly, as $''\alpha''$ increases the error is reduced for $\beta = 0.5$. Also, errors coincide for $\beta = 0$ and $\beta = 1$. As in the classical methods, the error can be further minimized by reducing the width size.

Case (2):. Presume that $\tilde{y}(t)$ is H_2- differentiable in the second form.

$\therefore$Equation (26) $\Rightarrow$

$$\left[y'_{lU}, y'_l, y'_{lL}, y', y'_{rL}, y'_r, y'_{rU}\right] = \left[-y_{lU}, -y_l, -y_{lL}, -y, -y_{rL}, -y_r, -y_{rU}\right] \tag{37}$$

Figure 1
Approximations - H_2- differentiable in the first form

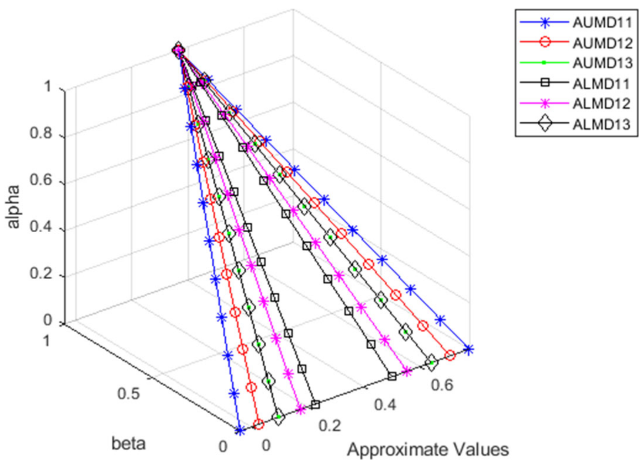
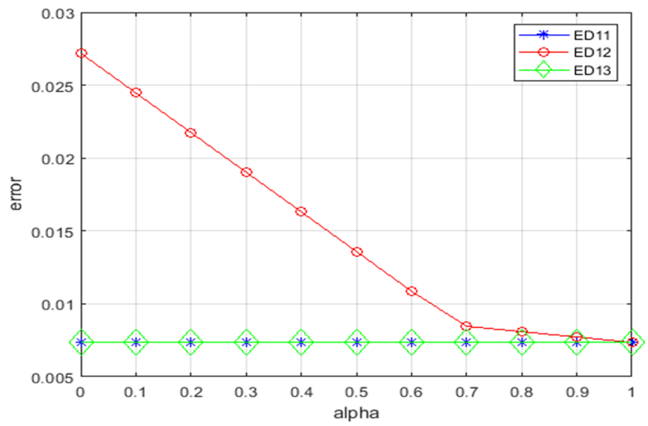


Figure 2
Errors at t = 1 - H_2- differentiable in the first form



Equation (37) gives three systems of ordinary differential equations given by:

$$\left.\begin{array}{l} y'_{lU} = -y_{lU} \\ y'_{rU} = -y_{rU} \end{array}\right\} \tag{38}$$

with initial conditions: $y_{0lU} = 0.85 + 0.15\alpha + 0.05\beta - 0.05\alpha\beta$; $y_{0rU} = 1.15 - 0.15\alpha - 0.05\beta + 0.05\alpha\beta$.

$$\left.\begin{array}{l} y'_l = -y_l \\ y'_r = -y_r \end{array}\right\} \tag{39}$$

with initial conditions: $y_{0l} = 0.9 + 0.1\alpha$; $y_{0r} = 1.1 - 0.1\alpha$. And

$$\left.\begin{array}{l} y'_{lL} = -y_{lL} \\ y'_{rL} = -y_{rL} \end{array}\right\} \tag{40}$$

with the initial conditions: $y_{0lL} = 0.94 + 0.06\alpha - 0.04\beta + 0.04\alpha\beta$; $y_{0rL} = 1.04 - 0.04\alpha + 0.06\beta - 0.06\alpha\beta$.

The analytical solution of UMF given in Equation (38) is

$$y_{\text{lU}} = (0.85 + 0.15\alpha + 0.05\beta - 0.05\alpha\beta)e^{-t};$$
$$y_{\text{rU}} = (1.15 - 0.15\alpha - 0.05\beta + 0.05\alpha\beta)e^{-t}.$$

The analytical solution of the principal MF given by Equation (39) is

$$y_{\text{l}} = (0.90 + 0.10\alpha)e^{-t};\ y_{\text{r}} = (1.10 - 0.10\alpha)e^{-t}.$$

The analytical solution of LMF given by Equation (39) is

$$y_{\text{lL}} = (0.94 + 0.06\alpha - 0.04\beta + 0.04\alpha\beta)e^{-t};$$
$$y_{\text{rL}} = (1.04 - 0.04\alpha + 0.06\beta - 0.06\alpha\beta)e^{-t}.$$

By fixing $\beta = 0$, permitting $''\alpha''$ to vary in [0, 1], the approximate solutions of UMF (AUMD21) and LMF (ALMD21) are obtained by the method proposed in Section 4 (Runge–Kutta method of order 4) for $''t = 1''$ with $h = 0.1$.

Similarly, the approximate solutions of UMF (AUMD22, AUMD23) and LMF (ALMD22, ALMD23) are simulated for $\beta = 0.5$ and $\beta = 1$. All these approximations are plotted in Figure 3.

The errors (ED21, ED22, and ED23) between the approximate solutions and exact solutions are plotted in Figure 4.

Figure 3
Approximations - H_2- differentiable in the second form

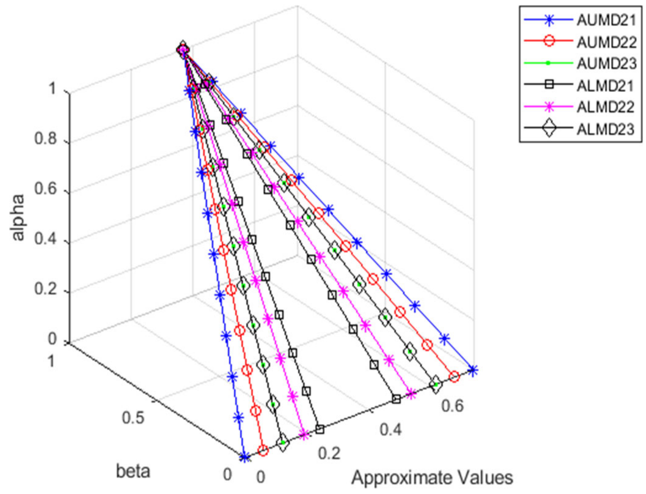
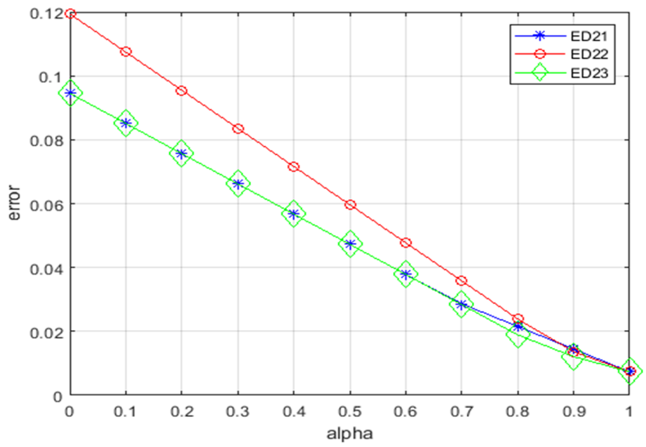


Figure 4
Errors at t = 1 - H_2- differentiable in the second form



Clearly, as $''\alpha''$ increases the error is reduced. Also, errors coincide for $\beta = 0$ and $\beta = 1$.

As in the classical methods, the error can be further minimized by reducing the step size.

Example 5.2. Consider a T2FIV problem $\widetilde{y}'(t) = 5 - \widetilde{y}(t)$; $t \in [0, 1]$, with the initial condition

$$\widetilde{y}(t_0) = \widetilde{y}_0 = [y_{0lU}, y_{0l}, y_{0lL}, y_0, y_{0rL}, y_{0r}, y_{0rU}] = [7, 8, 9, 10, 11, 12, 13] \quad (41)$$

(i.e.) $y_{0lU} = 7$; $y_{0l} = 8$; $y_{0lL} = 9$; $y_0 = 10$; $y_{0rL} = 11$; $y_{0r} = 12$; $y_{0rU} = 13$.

The parametric forms of the initial conditions are given by:

$$y_{0lU} = 7 + 3\alpha + \beta - \alpha\beta;$$

$$y_{0l} = 8 + 2\alpha;$$

$$y_{0lL} = 9 + \alpha - \beta + \alpha\beta;$$

$$y_{0rL} = 11 - \alpha + \beta - \alpha\beta;$$

$$y_{0r} = 12 - 2\alpha;$$

$$y_{0rU} = 13 - 3\alpha - \beta + \alpha\beta..$$

Now, let $\widetilde{y}(t) = [y_{lU}, y_l, y_{lL}, y, y_{rL}, y_r, y_{rU}]$ and hence $\widetilde{y}'(t) = [y'_{lU}, y'_l, y'_{lL}, y', y'_{rL}, y'_r, y'_{rU}]$

$\therefore$ Equation (25) $\Rightarrow$

$$[y'_{lU}, y'_l, y'_{lL}, y', y'_{rL}, y'_r, y'_{rU}] = 5 - [y_{lU}, y_l, y_{lL}, y, y_{rL}, y_r, y_{rU}] \quad (42)$$

Case (1): Presume that $\widetilde{y}(t)$ is H_2- differentiable in the first form.

The analytical solution of UMF of the T2FIV given in Equation (42) is

$$y_{\mathrm{lU}} = -(3 - 3\alpha - \beta + \alpha\beta)e^t + 5e^{-t} + 5;$$

$$y_{\mathrm{rU}} = (3 - 3\alpha - \beta + \alpha\beta)e^t + 5e^{-t} + 5.$$

The analytical solution of the principal MF is

$$y_{\mathrm{l}} = -(2 - 2\alpha)e + 5e^{-t} + 5;$$

$$y_{\mathrm{r}} = (2 - 2\alpha)e^t + 5e^{-t} + 5..$$

The analytical solution of LMF is

$$y_{\mathrm{lL}} = -(1 - \alpha + \beta - \alpha\beta)e^t + 5e^{-t} + 5;$$

$$y_{\mathrm{rL}} = (1 - \alpha + \beta - \alpha\beta)e^t + 5e^{-t} + 5..$$

The analytical and approximate solutions are obtained for $t = 1$ with step size $h = 0.1$.

Errors between analytical and approximate solutions of UMFs for $''\beta = 0''$; $\beta = 0.5$, and $\beta = 1$ are listed in Table 1.

Errors between analytical and approximate solutions of LMFs for $''\beta = 0''$; $\beta = 0.5$, and $\beta = 1$ are listed in Table 2.

The errors in both UMFs and LMFs can be reduced by increasing the step size.

Case (2): Presume that $\widetilde{y}(t)$ is H_2- differentiable in the second form.

Table 1
UMFs -Errors at t = 1 - H_2- differentiable in the first form

α	Error $(\beta = 0)$	Error $(\beta = 0.5)$	Error $(\beta = 1)$
0.0	14.33475708	11.9456309	9.556504716
0.1	12.90128137	10.75106781	8.600854244
0.2	11.46780566	9.556504716	7.645203773
0.3	10.03432995	8.361941627	6.689553302
0.4	8.600854244	7.167378538	5.73390283
0.5	7.167378538	5.972815447	4.778252359
0.6	5.73390283	4.778252359	3.822601887
0.7	4.300427122	3.583689268	2.866951414
0.8	2.866951414	2.389126178	1.911300943
0.9	1.433475707	1.19456309	0.955650472
1.0	3.332E-06	3.332E-06	3.332E-06

Table 2
LMFs- Errors at t = 1 - H_2- differentiable in the first form

α	Error $(\beta = 0)$	Error $(\beta = 0.5)$	Error $(\beta = 1)$
0.0	4.778252359	7.167378538	9.556504716
0.1	4.300427122	6.450640683	8.600854244
0.2	3.822601887	5.73390283	7.645203773
0.3	3.34477665	5.017164976	6.689553302
0.4	2.866951414	4.300427122	5.73390283
0.5	2.389126178	3.583689268	4.778252359
0.6	1.911300943	2.866951414	3.822601887
0.7	1.433475707	2.150213562	2.866951414
0.8	0.955650472	1.433475707	1.911300943
0.9	0.477825236	0.716737854	0.955650472
1.0	3.332E-06	3.332E-06	3.332E-06

Table 4
LMFs -Errors at t = 1 - H_2- differentiable in the second form

α	Error $(\beta = 0)$	Error $(\beta = 0.5)$	Error $(\beta = 1)$
0.0	3.332E-06	3.332E-06	3.332E-06
0.1	3.332E-06	3.334E-06	3.332E-06
0.2	3.333E-06	3.333E-06	3.332E-06
0.3	3.332E-06	3.332E-06	3.333E-06
0.4	3.332E-06	3.332E-06	3.333E-06
0.5	3.333E-06	3.332E-06	3.332E-06
0.6	3.333E-06	3.332E-06	3.333E-06
0.7	3.332E-06	3.334E-06	3.332E-06
0.8	3.332E-06	3.332E-06	3.333E-06
0.9	3.333E-06	3.332E-06	3.332E-06
1.0	3.332E-06	3.332E-06	3.332E-06

The analytical solution of UMF of the T2FIV given in Equation (42) is

$$y_{\mathrm{lU}} = (2 + 3\alpha + \beta - \alpha\beta)e^{-t} + 5;\ y_{\mathrm{rU}} = (8 - 3\alpha - \beta + \alpha\beta)e^{-t} + 5.$$

The analytical solution of the principal MF is: $y_{\mathrm{l}} = (3 + 2\alpha)e^{-t} + 5;\ y_{\mathrm{r}} = (7 - 2\alpha)e^{-t} + 5$.
The analytical solution of LMF is

$$y_{\mathrm{lL}} = (4 + \alpha - \beta + \alpha\beta)e^{-t} + 5;\ y_{\mathrm{rL}} = (6 - \alpha + \beta - \alpha\beta)e^{-t} + 5.$$

The analytical and approximate solutions are obtained for $t = 1$ with step size $h = 0.1$.

Errors between analytical and approximate solutions of UMFs for $''\beta = 0''$; $\beta = 0.5$, and $\beta = 1$ are listed in Table 3.

Errors between analytical and approximate solutions of LMFs for $''\beta = 0''$; $\beta = 0.5$, and $\beta = 1$ are listed in Table 4.

From Tables 3 and 4, it is found that there are no significant differences among the analytical and approximate solutions of UMF and LMF.

It is evident that the proposed method works well for the second kind of differentiability for the problems that we have considered for our study.

In this section, two T2FIV problems are considered and both the differentiability concepts have been applied to them.

Table 3
UMFs -Errors at t = 1 - H_2- differentiable in the second form

	Error $(\beta = 0)$	Error $(\beta = 0.5)$	Error $(\beta = 1)$
0.0	3.332E-06	3.332E-06	3.332E-06
0.1	3.332E-06	3.332E-06	3.332E-06
0.2	3.332E-06	3.332E-06	3.332E-06
0.3	3.333E-06	3.333E-06	3.333E-06
0.4	3.333E-06	3.333E-06	3.333E-06
0.5	3.332E-06	3.332E-06	3.332E-06
0.6	3.333E-06	3.333E-06	3.333E-06
0.7	3.332E-06	3.332E-06	3.332E-06
0.8	3.333E-06	3.333E-06	3.333E-06
0.9	3.332E-06	3.332E-06	3.332E-06
1.0	3.332E-06	3.332E-06	3.332E-06

In both the problems, the exact solution obtained by the first differentiability concept contains the term $''e^{t}''$ which tends to $''\infty''$ as $''t \to \infty''$. So, the two problems have unbounded solutions as $''t''$ increases.

On the other hand, both the problems under second differentiability have bounded solutions as $''t \to \infty''$.

Hence, depending upon the nature of the problem, we can forecast by our insight what will happen in the long run, and appropriate differentiability concept can be selected.

6. Conclusion

There are lots of circumstances in which the membership value of an element is not crisp. In these cases, T2FS plays a very important role. In this study, two differential equations with interval-valued T2FNs as its initial conditions are considered, and T2FIV problems are written in parametric forms. In each case of the differentiability, three sets of ordinary differential equations are obtained and these are assigned to UMF, primary MF, and LMF, respectively. The classical Runge–Kutta method of order 4 is applied to simulate the estimated solutions of the UMFs, the primary MFs, and the LMFs. These estimated solutions are compared with the respective analytical solutions for different values of $''\alpha''$ and $''\beta''$. The proposed method works better for the ordinary differential equations that are derived from the second form of differentiability concept than for those equations derived from the first form of the differentiability concept. It is obvious that the proposed method (a single step method) is competent than the classical Euler's and modified Euler's methods, and small step size can minimize the inaccuracy between the estimated solutions and analytical solutions. In future, multi-step methods will be used to explore the numerical results of T2FDEs. Further, the appropriate method of differentiability can be selected by analyzing the nature of the problems that have been considered for the study.

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Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

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