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Application of the Techniques of Determinants to Utility Maximization for a 4-Period Age-Specific Inter-Temporal Budget Constraint

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Abstract: In the modern economics, one has to choose financial strategies for investment and expenditure to optimize the expected utility over the subsequent lifetimes. In this article, the Lagrange multiplier method was used to derive a mathematical formulation to work out an optimal solution for a 4-period overlapping generation model with autonomous consumption to maximize a lifetime utility for households subject to age-specific inter-temporal budget constraints. Also, the Cramer's rule is used in finding the critical point where utility is maximized. Further test for sufficient conditions has been carried out using the Hessian determinants to check if there is a local maximum in the critical point C, where the utility is maximized. The inter-temporal marginal rate of substitution was implored to show the future growth path of utility maximization, and analytical argument was used to support such finding.

Keywords: household, inter-temporal, Lagrange multiplier, utility maximization, optimal solution

1. Introduction

The procedure of Lagrange multipliers (LMs) is a very useful method used in the determination of Cramer's rule solution and Hessian through the use of determinants under calculus. Islam examined a situation of an individual maximizing utility in two products subject to a budget constraint and considered the behavior of an individual end user, furnishing preference relations Islam (1997). Moolio (2002) extended the work of Islam (1997) to *n* products and used the method of calculus to obtain the requisite and adequate conditions, as well as taking into account Cobb–Douglas production function of two variables (capital and labor) and using the tool of LM method. He studied the behavior of a competitive firm by considering a cost minimization subject to an output constraint, see also, Moolio & Islam (2008) and Moolio et al. (2009). Typically considering an overlapping generation model is one of the models of economic growth that specifically analyzes the behavior of household, that is, on the behaviors and decision made by individuals and households regarding utility maximization (UM). The model was first proposed by Samuelson (1958); he introduced a micro founded model with heterogeneous agents. It manifests in two coexisting generations at each point in time: firstly, the agents are heterogeneous in terms of age, that is, young and old. This means that they have differences in terms of economic decision making. In addition, individuals have finite lifetime after which they die. This is a reality in most cases of individual life time. An alternative to this proposition was presented by Diamond (1965) and Blanchard (1985). Their procedure has been celebrated, since it allows for more than two periods within a life cycle and leads to simple analysis.

Therefore, infinitely lived agent models have become very important in economics because they can easily trace the decision an agent makes, if he wants to live forever or if he wants to live for few years. Overlapping generation model is good in answering questions that relate to policies, which affects different age cohorts as to what decision an individual or household make to maximize their utility. The model that will be presented here carries out the analysis in a discrete time, that is, we considered periods, t, $t+1$, $t+2$, etc. Overlapping generation model has been developed to account for complex economic interactions involving more than one generation. Economic models that incorporate demographic transition have the potential to enhance predictions of actual models. The framework presented by Diamond (1965) and Auerbach & Kotlikoff (1987) has popularized overlapping generation models, due to their ability to make predictions about important variables such as rate of returns on assets and outcomes of pension restructuring. The basic mechanism of overlapping generation model is driven by the life choices of representative individuals regarding education, labor supply, savings, consumption, and retirement. Such a setup permits the model to project the accumulation and transfer of wealth over time and across generations. More complex models involving multiple generations of individuals with heterogeneous life choice preferences can potentially reproduce large movements in assets prices and interest rate Huffman (1987).

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The concept of UM has been widely studied and applied in many facets of human existence; Aleskerov et al. (2007) have written a comprehensive book on the subject matter incorporating choice and preferences. Hu et al. (2005) considered the UM in incomplete markets taking small traders problems in the financial markets. The study separated exponential utility from the power and logarithmic cases. Palomar & Chiang (2006) presented a tutorial on the decomposition methods for the network UM. Herrnstein et al. (1993) studied internalities present in individual choice and incorporated the UM with milioration. The UM has been used in task scheduling Huang et al. (2021), in a multi-server system Karakoc et al. (2022), in data scheduling based on drone-assisted vehicular networks Fan et al. (2021), in wireless sensor network Chen et al. (2021), in intelligent computing Baktayan & Al-Balta (2021), and in online network analysis and cloud data centers for evolutionary multi-objective join customer services. Cao et al. (2022); Goudarzi et al. (2021). Ghasemi et al. (2022) showed that a humanitarian relief logistic network model can be used to locate shelters and distribution centers, determine routes, and allocate resources in uncertain and real-life disaster situations. The findings are that the Non-Dominated Sorting Genetic Algorithm (NSGA-II) is efficient and reliable for small and medium scale problems, with a maximum mean error of 0.63%. Khanchehzarrin et al. (2021) presented a new mixed integer nonlinear programming model for the time-dependent vehicle routing problem with time windows and intelligent travel times, and the model presented in the article leads to a 32% reduction in costs. In the area of pension reforms, Buyse et al. (2017) opined that a pay-as-you-go (PAYG) pension system with high weight on labor income for older individuals is preferred. This implies that a PAYG system advocated in Buyse et al. (2013) is more efficient than a fully funded private system, but it imposes significant welfare losses on low ability individuals, see also, Bucciol & Beetsma (2011). For more on UM in other areas, see also, Kudrna et al. (2015), Muto et al. (2012), Song & Yang (2010), and Ahmadi & Ghasemi (2022).

In this article therefore, the problem of maximization of utility in a 4-period overlapping generation model with autonomous consumption subject to age-specific budget constraints was examined. Baxley & Moorhouse (1984) proposed this problem in their article entitled "Lagrange Multiplier Problems in Economics". In Section 2, consequent on the works of Islam et al. (2009) and Khanchehzarrin et al. (2021), we construct the mathematical model for the problem and obtain the Cramer's rule solution using the techniques of determinant to find the critical points were utility is maximized optimally for household. In the context of this particular model formulation, we derive and test for the sufficient conditions using the bordered Hessian determinants stated in Section 3. In Section 4, we considered the inter-temporal marginal rate of substitution considering the extent to which consumption at a particular time will be exchanged for another period, following Islam et al. (2009). We discuss the optimal solution results, inspecting the behavior of an individual household. In the last Section 5, concluding and final remarks are given.

2. Methodology

2.1. The mathematical model

Since we have confine ourselves to 4-period overlapping generation model with age-specific constraints, we then form the individual household objective utility function $Max(U_t)$ which must be maximized subject to the age-specific constraints.

$$Max(U_t) = \frac{C_{0,t}^{1-\theta}}{1-\theta} + \beta\frac{C_{1,t+1}^{1-\theta}}{1-\theta} + \beta^2\frac{C_{2,t+2}^{1-\theta}}{1-\theta} + \beta^3\frac{C_{3,t+3}^{1-\theta}}{1-\theta} \tag{1}$$

The four age-specific inter-temporal budget constraints are as follows:

$$c_{0,t} = w_t - k_{0,t+1} \tag{2}$$

$$c_{1,t+1} = w_{t+1} + (1+r_{t+1})k_{0,t+1} - k_{1,t+2} \tag{3}$$

$$c_{2,t+2} = w_{t+2} + (1+r_{t+2})k_{1,t+2} - k_{2,t+3} \tag{4}$$

$$c_{3,t+3} = (1+r_{t+3})k_{2,t+3} \tag{5}$$

The household UM problem can then be expressed by forming the Lagrange function Γ as:

$$\begin{aligned}\Gamma &= \frac{C_{0,t}^{1-\theta}}{1-\theta} + \beta\frac{C_{1,t+1}^{1-\theta}}{1-\theta} + \beta^2\frac{C_{2,t+2}^{1-\theta}}{1-\theta} + \beta^3\frac{C_{3,t+3}^{1-\theta}}{1-\theta}\\ &+\lambda\frac{(1+r_{t+1})(1+r_{t+2})w_t+(1+r_{t+2})w_{t+1}+w_{t+2}}{(1+r_{t+1})(1+r_{t+2})}\\ &-\frac{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})c_{0,t}-(1+r_{t+2})(1+r_{t+3})c_{1,t+1}-(1+r_{t+3})c_{2,t+2}-c_{3,t+3}}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})}\end{aligned} \tag{6}$$

Setting the partial derivatives of equation (6) equal to zero, we get the following first-order necessary conditions for maximization:

$$\frac{\partial\Gamma}{\partial c_{0,t}} = c_{0,t}^{-\theta} - \lambda = 0 \tag{7}$$

$$\frac{\partial\Gamma}{\partial c_{1,t+1}} = \beta c_{0,t}^{-\theta} - \frac{\lambda}{(1+r_{t+1})} = 0 \tag{8}$$

$$\frac{\partial\Gamma}{\partial c_{2,t+2}} = \beta^2 c_{2,t+2}^{-\theta} - \frac{\lambda}{(1+r_{t+1})(1+r_{t+2})} = 0 \tag{9}$$

$$\frac{\partial\Gamma}{\partial c_{3,t+3}} = \beta^3 c_{3,t+3}^{-\theta} - \frac{\lambda}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} = 0 \tag{10}$$

$$\begin{aligned}\frac{\partial\Gamma}{\partial\lambda} &= \frac{(1+r_{t+1})(1+r_{t+2})w_t+(1+r_{t+2})w_{t+1}+w_{t+2}}{(1+r_{t+1})(1+r_{t+2})}\\ &-\frac{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})c_{0,t}-(1+r_{t+2})(1+r_{t+3})c_{1,t+1}-(1+r_{t+3})c_{2,t+2}-c_{3,t+3}}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} = 0\end{aligned} \tag{11}$$

We implored the use of determinant and Cramer's rule to essentially obtain an optimal solution to the maximization problem. First of all, we obtain the determinant of the coefficient matrix followed by the determinant of the special matrix formed from the original coefficient matrix by replacing the column of coefficients with the column vector.

$$|A| = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & \beta & 0 & 0 & \frac{-1}{(1+r_{t+1})} \\ 0 & 0 & \beta^2 & 0 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} \\ 0 & 0 & 0 & \beta^3 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} \\ 1 & \frac{-1}{(1+r_{t+1})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} & 0 \end{bmatrix}$$
$$\begin{bmatrix} C_{0,t}^{-\theta} \\ C_{1,t+1}^{-\theta} \\ C_{2,t+2}^{-\theta} \\ C_{3,t+3}^{-\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ R \end{bmatrix} \tag{12}$$

where

$$R = \frac{-(1+r_{t+1})(1+r_{t+2})w_t - (1+r_{t+2})w_{t+1} - w_{t+2}}{(1+r_{t+1})(1+r_{t+2})}$$

First, expanding the matrix along the first and the last columns, the determinant can be obtained as follows:

$$|A| = -\beta\left[\frac{1}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})^2} + \frac{\beta}{(1+r_{t+1})^2(1+r_{t+2})^2} + \frac{\beta^2}{(1+r_{t+1})^2} - \beta^3\right] \tag{13}$$

Next is to find $|A_1|$,

$$|A_1| = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & \beta & 0 & 0 & \frac{-1}{(1+r_{t+1})} \\ 0 & 0 & \beta^2 & 0 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} \\ 0 & 0 & 0 & \beta^3 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} \\ R & \frac{-1}{(1+r_{t+1})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} & 0 \end{bmatrix} \begin{bmatrix} C_{0,t}^{-\theta} \\ C_{1,t+1}^{-\theta} \\ C_{2,t+2}^{-\theta} \\ C_{3,t+3}^{-\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ R \end{bmatrix} \tag{14}$$

In the above matrix (equation (14)), it shows that all the compartments will vanish and the result of the determinant will give zero. That is,

$$|A_1| = 0 \tag{15}$$

Similarly,

$$|A_2| = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \frac{-1}{(1+r_{t+1})} \\ 0 & 0 & \beta^2 & 0 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} \\ 0 & 0 & 0 & \beta^3 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} \\ 1 & R & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} & 0 \end{bmatrix} \begin{bmatrix} C_{0,t}^{-\theta} \\ C_{1,t+1}^{-\theta} \\ C_{2,t+2}^{-\theta} \\ C_{3,t+3}^{-\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ R \end{bmatrix} \tag{16}$$

Considering the first and last columns, the determinant is

$$|A_2| = -\beta^5\left[\frac{(1+r_{t+1})(1+r_{t+2})w_t + (1+r_{t+2})w_{t+1} + w_{t+2}}{(1+r_{t+1})^2(1+r_{t+2})}\right] \tag{17}$$

Next,

$$|A_3| = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & \beta & 0 & 0 & \frac{-1}{(1+r_{t+1})} \\ 0 & 0 & 0 & 0 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} \\ 0 & 0 & 0 & \beta^3 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} \\ 1 & \frac{-1}{(1+r_{t+1})} & R & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} & 0 \end{bmatrix} \begin{bmatrix} C_{0,t}^{-\theta} \\ C_{1,t+1}^{-\theta} \\ C_{2,t+2}^{-\theta} \\ C_{3,t+3}^{-\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ R \end{bmatrix} \tag{18}$$

To obtain the determinant, only the first and last columns were considered, because the rest of the columns will vanish. Therefore,

$$|A_3| = -\beta^4\left[\frac{(1+r_{t+1})(1+r_{t+2})w_t + (1+r_{t+2})w_{t+1} + w_{t+2}}{(1+r_{t+1})^2(1+r_{t+2})^2}\right] \tag{19}$$

Next,

$$|A_4| = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & \beta & 0 & 0 & \frac{-1}{(1+r_{t+1})} \\ 0 & 0 & \beta^2 & 0 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} \\ 0 & 0 & 0 & 0 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} \\ 1 & \frac{-1}{(1+r_{t+1})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} & R & 0 \end{bmatrix} \begin{bmatrix} C_{0,t}^{-\theta} \\ C_{1,t+1}^{-\theta} \\ C_{2,t+2}^{-\theta} \\ C_{3,t+3}^{-\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ R \end{bmatrix} \tag{20}$$

Here, we also expand in terms of the first and last columns to get,

$$|A_4| = -\beta^3\left[\frac{(1+r_{t+1})(1+r_{t+2})w_t + (1+r_{t+2})w_{t+1} + w_{t+2}}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})}\right] \tag{21}$$

Next is to obtain the determinant of A_5,

$$|A_5| = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta^2 & 0 & 0 \\ 0 & 0 & 0 & \beta^3 & 0 \\ 1 & \frac{-1}{(1+r_{t+1})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} & R \end{bmatrix} \begin{bmatrix} C_{0,t}^{-\theta} \\ C_{1,t+1}^{-\theta} \\ C_{2,t+2}^{-\theta} \\ C_{3,t+3}^{-\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ R \end{bmatrix} \tag{22}$$

Expanding for only the first column means that the other columns will vanish.

Therefore,

$$|A_5| = -\beta^6\left[\frac{(1+r_{t+1})(1+r_{t+2})w_t + (1+r_{t+2})w_{t+1} + w_{t+2}}{(1+r_{t+1})(1+r_{t+2})}\right] \tag{23}$$

Now, obtain the Cramer's rule solution by dividing equations (15), (17), (19), (21), and (23) each by equation (13) to get the critical values where utility is at its best maximized.

$$\begin{aligned} C_{0,t}^{-\theta} &= \frac{|A_1|}{|A|} \\ &= \frac{0}{-\beta\left[\frac{1}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})^2}+\frac{\beta}{(1+r_{t+1})^2(1+r_{t+2})^2}+\frac{\beta^2}{(1+r_{t+1})^2}-\beta^3\right]} \\ &= 0 \end{aligned} \tag{24}$$

$$\begin{aligned} C_{1,t+1}^{-\theta} &= \frac{|A_2|}{|A|} \\ &= \frac{-\beta^5\left[\frac{(1+r_{t+1})(1+r_{t+2})w_t+(1+r_{t+2})w_{t+1}+w_{t+2}}{(1+r_{t+1})^2(1+r_{t+2})}\right]}{-\beta\left[\frac{1}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})^2}+\frac{\beta}{(1+r_{t+1})^2(1+r_{t+2})^2}+\frac{\beta^2}{(1+r_{t+1})^2}-\beta^3\right]} \end{aligned} \tag{25}$$

$$\begin{aligned} C_{2,t+2}^{-\theta} &= \frac{|A_3|}{|A|} \\ &= \frac{-\beta^4\left[\frac{(1+r_{t+1})(1+r_{t+2})w_t+(1+r_{t+2})w_{t+1}+w_{t+2}}{(1+r_{t+1})^2(1+r_{t+2})^2}\right]}{-\beta\left[\frac{1}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})^2}+\frac{\beta}{(1+r_{t+1})^2(1+r_{t+2})^2}+\frac{\beta^2}{(1+r_{t+1})^2}-\beta^3\right]} \end{aligned} \tag{26}$$

$$\begin{aligned} C_{3,t+3}^{-\theta} &= \frac{|A_4|}{|A|} \\ &= \frac{-\beta^3\left[\frac{(1+r_{t+1})(1+r_{t+2})w_t+(1+r_{t+2})w_{t+1}+w_{t+2}}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})}\right]}{-\beta\left[\frac{1}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})^2}+\frac{\beta}{(1+r_{t+1})^2(1+r_{t+2})^2}+\frac{\beta^2}{(1+r_{t+1})^2}-\beta^3\right]} \end{aligned} \tag{27}$$

$$\begin{aligned} \lambda &= \frac{|A_5|}{|A|} \\ &= \frac{-\beta^6\left[\frac{(1+r_{t+1})(1+r_{t+2})w_t+(1+r_{t+2})w_{t+1}+w_{t+2}}{(1+r_{t+1})(1+r_{t+2})}\right]}{-\beta\left[\frac{1}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})^2}+\frac{\beta}{(1+r_{t+1})^2(1+r_{t+2})^2}+\frac{\beta^2}{(1+r_{t+1})^2}-\beta^3\right]} \end{aligned} \tag{28}$$

Therefore in principle, equations (24), (25), (26), (27), and (28), respectively, give us the optimal solution where utility is maximized and each value being a function of the parameters $C_{0,t}$, $C_{1,t+1}$, $C_{2,t+2}$, $C_{3,t+3}$.

2.2. Second-order sufficient condition

This condition is used to ensure that the optimal solution obtained is at its maximum. We therefore check if the solutions obtained in equations (24), (25), (26), (27), and (28) are the critical points for the maximum problem. In doing this, the optimal solution using the bordered principal minors of the bordered Hessian is tested,

$$|\bar{H}| = \begin{bmatrix} -\theta c_{0,t}^{-\theta-1} & 0 & 0 & 0 & -1 \\ 0 & -\theta\beta c_{1,t+1}^{-\theta-1} & 0 & 0 & \frac{-1}{(1+r_{t+1})} \\ 0 & 0 & -\theta\beta^2 c_{2,t+2}^{-\theta-1} & 0 & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} \\ 0 & 0 & 0 & -\theta\beta^3 c_{3,t+3}^{-\theta-1} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} \\ 1 & \frac{-1}{(1+r_{t+1})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})} & \frac{-1}{(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})} & 0 \end{bmatrix} \tag{29}$$

$$|H_1| = -\theta c_{0,t}^{-\theta-1} < 0 \tag{30}$$

This shows that the value of $|H_1|$ is negative and less than zero. We then evaluate $|H_2|$ by taking the bordered Hessian matrix which is a 4×4 matrix.

Therefore,

$$|H_2| = \theta^4\beta^6 c_{1,t+1}^{-\theta-1} c_{2,t+2}^{-\theta-1} c_{3,t+3}^{-\theta-1} > 0 \tag{31}$$

This shows that the value of $|H_1|$ and $|H_2|$ alternates, since the result of $|H_2|$ is positive and greater than zero. We now evaluate for $|H_3|$ which is equal to the matrix of $|\bar{H}|$. Here, we consider only the extreme column of $|\bar{H}|$ since the other columns vanish. Therefore,

$$\begin{aligned} |H_3| &= \theta^3\beta^3\left[\frac{c_{0,t}^{-\theta-1}c_{1,t+1}^{-\theta-1}c_{2,t+2}^{-\theta-1}+(1+r_{t+3})^2\beta c_{0,t}^{-\theta-1}c_{1,t+1}^{-\theta-1}c_{3,t+3}^{-\theta-1}}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})^2}\right] \\ &\quad +\theta^3\beta^5\left[\frac{c_{0,t}^{-\theta-1}c_{2,t+2}^{-\theta-1}c_{3,t+3}^{-\theta-1}+(1+r_{t+1})^2\beta c_{1,t+1}^{-\theta-1}c_{2,t+2}^{-\theta-1}c_{3,t+3}^{-\theta-1}}{(1+r_{t+1})^2}\right] \end{aligned} \tag{32}$$

For $|H_3| = |\bar{H}|$ to alternate with the previous $|H_1|$ and $|H_2|$, we set the condition that

$$\begin{aligned} |H_3| &= \theta^3\beta^3\left[\frac{c_{0,t}^{-\theta-1}c_{1,t+1}^{-\theta-1}c_{2,t+2}^{-\theta-1}+(1+r_{t+3})^2\beta c_{0,t}^{-\theta-1}c_{1,t+1}^{-\theta-1}c_{3,t+3}^{-\theta-1}}{(1+r_{t+1})^2(1+r_{t+2})^2(1+r_{t+3})^2}\right] \\ &< \theta^3\beta^5\left[\frac{c_{0,t}^{-\theta-1}c_{2,t+2}^{-\theta-1}c_{3,t+3}^{-\theta-1}+(1+r_{t+1})^2\beta c_{1,t+1}^{-\theta-1}c_{2,t+2}^{-\theta-1}c_{3,t+3}^{-\theta-1}}{(1+r_{t+1})^2}\right] \end{aligned} \tag{33}$$

Hence, $|H_3| = |\bar{H}| < 0$.

Since the signs of $|H_1|$, $|H_2|$, and $|H_3|$ conservatively alternate, we conclude that the optimal solution obtained in equation (12) is at its maximum when we evaluated the necessary condition and sufficient conditions for UM.

2.3. Inter-temporal marginal rate of substitution

Considering the 4-period generation of UM of household, the decision made by household at each period regarding its consumption will affect its consumption positively or negatively in the future. The inter-temporal marginal rate of substitution defines the amount of consumption available in a particular time that will be exchanged for a small amount of consumption available in another time period. We will use the additive inter-temporal utility for any pair of consumption flow of two periods. The inter-temporal marginal utility is defined as:

$$\partial\Gamma = [c_{0,t}^{-\theta}]\partial c_{0,t} + [(1+r_{t+1})\beta c_{1,t+1}^{-\theta}]\partial c_{1,t+1} = 0 \tag{34}$$

The inter-temporal marginal rate of substitution ($IMRS$) of consumption in these two periods t and $t+1$ of the time separable inter-temporal utility is equal to

$$IMRS_{\frac{t+1}{t}} = \frac{c_{1,t+1}}{c_{0,t}} = [(1+r_{t+1})\beta]^{\frac{1}{\theta}} \tag{35}$$

We will follow the same procedure to find the inter-temporal marginal utility by pairing the other periods.

$$\begin{aligned} \partial\Gamma &= [(1+r_{t+1})\beta c_{1,t+1}^{-\theta}]\partial c_{1,t+1} \\ &\quad + [(1+r_{t+1})(1+r_{t+2})\beta^2 c_{2,t+2}^{-\theta}]\partial c_{2,t+2} = 0 \end{aligned} \tag{36}$$

Therefore, the inter-temporal marginal rate of substitution of consumption in this two period $t+1$ and $t+2$ of the time separable inter-temporal utility is equal to

$$IMRS_{\frac{t+2}{t+1}} = \frac{c_{2,t+2}}{c_{1,t+1}} = [(1+r_{t+2})\beta]^{\frac{1}{\theta}} \tag{37}$$

Lastly, the inter-temporal marginal rate of substitution of consumption for the last pair will be

$$\partial\Gamma = [(1+r_{t+1})(1+r_{t+2})\beta^2 c_{2,t+2}^{-\theta}]\partial c_{2,t+2} + [(1+r_{t+1})(1+r_{t+2})(1+r_{t+3})\beta^3 c_{3,t+3}^{-\theta}]\partial c_{3,t+3} = 0 \tag{38}$$

Therefore, the inter-temporal marginal rate of substitution of consumption for these last two periods $t+2$ and $t+3$ of the time separable inter-temporal utility is equal to

$$IMRS_{\frac{t+3}{t+2}} = \frac{c_{3,t+3}}{c_{2,t+2}} = [(1+r_{t+3})\beta]^{\frac{1}{\theta}} \tag{39}$$

3. Results and Discussions

3.1. Optimal solution

From the analytical formulation, the method of LM has been helpful in deriving a mathematical formulation to obtain the optimal solution for the 4-period lived agent of overlapping generation model subject to 4-period age-specific inter-temporal budget constraints. The critical points obtained are points at which household maximizes its lifetime utility from the objective function. The optimal solution so obtained is also tested using the Hessian principal minors to determine if the optimal solution is at its maximum.

3.2. Inter-temporal marginal rate of substitution

The inter-temporal marginal rate of substitution obtained in (35), (37), and (39) shows that consumption of individuals when young, when in the middle age, and old will depend positively on interest rate r_{t+1}, r_{t+2}, and r_{t+3}, respectively. Consumption will also negatively depend on the time discount factor β. The greater is β, the greater is the individual's impatience and hence lowers his consumption in the future and vice versa. Similarly, the inter-temporal marginal rate of substitution also shows that the household consumption depends negatively on the risk aversion parameter θ. A θ that approaches 1 means that the utility will decline. It also follows that the marginal utility of consumption diminishes fast. This also tells us that such individual will have less to save and the amount of interest will always be very minimal as such consumption will also be relatively affected in future. We therefore conclude that it is always important for household to always make the right decision so that their consumption will not be affected negatively in the future time.

4. Conclusion

Household behavior toward edible consumption is a complex and complicated problem. It varies from one family to another depending on family size and lifestyle. This study investigated the UM and applied the approach of LM to maximize utility for a household in a 4-period generation subject to age-specific inter-temporal budget constraint; we derived the mathematical formulation to obtain the optimal solution for household. With the help of inter-temporal marginal rate of substitution, we observe that the behavior of a household, that is, its decision regarding its level of consumption at a particular period, affects its consumption positively or negatively in the future.

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The authors contributed equally in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

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