Received: 30 April 2024 | Revised: 2 July 2024 | Accepted: 29 July 2024 | Published online: 13 August 2024

RESEARCH ARTICLE

Journal of Computational and Cognitive Engineering
2024, Vol. 00(00) 1–8
DOI: 10.47852/bonviewJCCE42023298

BON VIEW PUBLISHING

Topological Analysis of Image Reconstruction Based on Polar Complex Exponential Transform

Manoj Kumar Singh[1], **Deepika Saini**[2] **and Sanoj Kumar**[3,*]

[1] *School of Computer Science Engineering and Technology, Bennett University, India*

[2] *Department of Mathematics, Graphic Era (Deemed to be) University, India*

[3] *Data Science Cluster, University of Petroleum and Energy Studies, India*

Abstract: Image moments are an important tool used for image reconstruction. An image can be represented in terms of image moments, which is known as image reconstruction from its moments. To construct an image from the moments, a question often arises that how many moments are required for the reconstruction of image. Theoretically, many image moments may be required for accurately reconstructing an image. However, since image moment construction can be computationally challenging, often in practice only a finite number of moments are used for the image reconstruction. The difference in reconstructed and original image for many numbers of images can lead to a satisfying answer. However, accurate reconstruction may not often be possible, and we are often relying on other approaches to find similarity between the original image and reconstructed image. We used a similarity based on a topological data analysis tool known as the persistence diagram, determining the bottleneck distance between the original and the reconstructed image as the measure of similarity. Our investigation was conducted on the common images utilized in image processing tasks. The findings indicate that there is no direct correlation between the number of moments and the quality of image reconstruction. It is necessary to choose an appropriate number of moments, which may be very small, instead of calculating a high number of image moments.

Keywords: image moments, Polar Harmonic Transform (PHT), topological data analysis, image reconstruction

1. Introduction

Sharing information is simpler than it has ever been in the modern world, yet this also makes it easier for people to spread false information. Sharing multimedia, which may include concealed watermarks, is a frequent practice; however, this practice may damage the message that was intended to be sent [1]. The use of robust watermarking techniques is crucial for the protection of copyright and the validation of sources since photos are transferred across channels that may or may not be secure. Even though there are a lot of strategies, only a few of them are resistant to rotation. Although numerous approaches, including DC coefficients and picture moments, have been investigated, there are still limits. Taking this into consideration, a unique computational approach for computing Polar Harmonic Transform (PHT) moments, which is presented in reference [2], offers promise for enhancing the accuracy of watermarking, particularly for aerial photos. Through the utilization of polar complex exponential transform (PCET) and PST [3, 4], these strategies improve the robustness of the system against rotating attacks while simultaneously preserving its reversibility. An examination of the proposed method in comparison to other approaches reveals that it possesses greater rotational invariance, hence showing the effectiveness of the proposed method.

It is well known that orthogonal circular moments are invariant in translation, rotation, and scaling. This makes the moments for image reconstruction an important tool, since many image descriptors are not rotational and scaling invariant. Image moments are essential for reconstructing images because they can capture important geometric and intensity features, reduce noise and artifacts, provide efficient data representation, improve algorithmic stability and convergence, allow real-time processing, and work well with machine learning methods. Due to their extensive and strong characteristics, they are essential for achieving image reconstructions that are of high quality and efficiency. These moments have a wide range of uses, including the detection of counterfeiting and the imaging of biological conditions. To compute these moments, it is necessary to integrate basis functions with image functions over a unit disk. This process sometimes necessitates mapping from rectangular to polar coordinates. Rotational invariance is best achieved by the utilization of techniques such as inscribed circle mapping. Object recognition and medical imaging are only two of the many industries that could benefit greatly from the application of image moments. They are defined as double integrals of a two-dimensional function with base polynomial functions, and they have applications in a variety of fields, including picture registration, object detection, and more. These moments may be geometric or orthogonal, with the

***Corresponding author:** Sanoj Kumar, Data Science Cluster, University of Petroleum and Energy Studies, India. Email: sanoj.kumar@ddn.upes.ac.in

latter providing robust computation in discrete domains. Both these types of moments are possible. Over particular domains, such as rectangles or unit disks, it is possible to define many types of orthogonal moments, such as the Legendre and Chebyshev moments.

Gridding methods are a significant factor in determining the accuracy of moment computation, with polar grids typically producing more accurate results. On the other hand, new research indicates that rectangular grids, when combined with Gaussian numerical integration, have the potential to beat polar grids in applications such as watermarking. While using conventional approaches, ambiguity occurs while integrating pixels that are located on the boundary of the unit circle, which results in approximations being developed. To circumvent this limitation, the strategy that has been proposed plans to make use of the complete unit disk for the computation of moments by employing a method that combines analytical and numerical approaches. The accuracy of moment computation is improved by this approach, which is beneficial to a variety of applications that deal with image processing. This study involved analyzing similarity between an image and a reconstructed image using a tool called bottleneck distance, which is derived from topological data analysis. The bottleneck distance, when used in conjunction with the persistence diagram, offers a precise measure of similarity between an image and its reconstructed counterpart. This metric, derived from the image's topology, additionally ensures that the image moments maintain the integrity of the image's rotation and scaling. Hence, the integration of topology and image moments will introduce a novel aspect to image processing and reconstruction.

2. Literature Review

This study presents a new two-stage robust reversible watermarking system. It uses the PHT [3, 4]. The purpose of this scheme is to achieve high robustness and capacity while simultaneously retaining invisibility and reversibility. Here, PHT moments are used for robust watermarking due to their resilience to attacks, computational efficiency, and numerical stability, especially when compared to other transformations like Fourier Mellin and Zernike moments. Watermarking uses PHT moments because of their precision and potential resistance to attacks [5]. There is a correlation between the total absolute values of the PHT moment's order, repetition, and robustness, indicating that these values are suitable for watermark embedding [6]. The PHT moments are the image's global properties, and they serve as a foundation for the development of robust watermark embedding schemes [7, 8]. The suggested transforms are better than the current PHTs and fractional orthogonal transforms in terms of accuracy, numerical stability, digital image reconstruction, RST invariances, noise resistance, and how quickly they can be used [9]. The research presents fractional-order PHTs for grayscale images [10], as well as fractional-order quaternion PHTs for color images [11]. We use both transforms to analyze grayscale photos. A computational framework based on kernels achieves an effective computation of the transforms in polar coordinates [12].

The work presented in Ali et al. [13] and Pal et al. [14] explains a method for categorizing textures. The approach makes use of picture visibility graphs and topological data analysis. We evaluated the method on two different image texture datasets and found encouraging results. This study suggests that there is potential for successfully integrating topological features with graph-based approaches for texture classification. When applied to the Brodatz dataset, the gradient boosting model managed to obtain the maximum level of accuracy. The KTH-TIPS dataset was the one in which the support vector machine model demonstrated the best level of accuracy.

Algebraic methods (AM) and PHT are two examples of image reconstruction techniques that play important roles in improving the quality and robustness of images. AM provides efficient field function determination by unique grid discretization algorithms, as explained in Qi et al. [15] and Qu [16]. This results in a reduction in the amount of computing burden and memory needed. Conversely, PHT, as emphasized in Yang and Deng [17], focuses on developing image-invariant descriptors for watermarking applications, thereby enhancing its resilience against attacks. Furthermore, the use of wavelet transforms and deep learning in laser polarization picture reconstruction, as demonstrated in Zhang et al. [18], demonstrates the development of innovative techniques for denoising, feature extraction, and edge detection, which ultimately results in enhanced visual effects. These varied approaches illustrate the value of creative methods in image processing, which can respond to a variety of requirements, including speed, accuracy, resilience, and visual quality.

A process that includes transforming images into polar representations to improve orientation invariance is known as image polar transformation [19]. This transformation method, which includes orientation-invariant histograms of oriented gradients [20], is effective in improving visual tasks such as facial expression recognition. However, methods for reconstructing images, like those used in snapshot-channeled imaging spectroscopy, use neural networks to get around the problems that come up with Fourier-based reconstructions. This ensures that the reconstruction's spectral and polarization accuracy are high [20]. Furthermore, they design polarization super-resolution approaches to enhance the detailed polarization information in images. Deep convolutional neural networks can efficiently balance intensity and polarization restoration for high-resolution polarized image reconstruction [21]. These techniques, when taken as a whole, contribute to the technological advancement of image processing and analysis in a variety of domains, ranging from remote sensing to microscopic super-resolution imaging.

The suggested orientation-invariant image representation, which uses polar models for both handcrafted features and deep learning features, is not only on par with the best methods but also keeps a compact representation on a set of difficult benchmark image datasets [22]. A vision transformer-based representation learning framework for PolSAR image classification is proposed [23]. This framework makes use of self-attention as an alternative to convolution, which shifts the focus from the information in local neighborhoods to the long-range interactions between each pixel. An adaptive image reconstruction approach based on variable exponential function regularization is proposed [24]. This method focuses on the diversity of the PSF and makes use of variable-exponent regularization to improve the kernel's flexibility. The purpose of this paper is to propose an alternative reconstruction algorithm that is based on the image recombination transform [25]. This algorithm offers an alternative solution to address this problem, even in the weak modulation depth of structured illumination microscopy. This solution is ideal for long-term, in vivo, super-resolution imaging of live cells and tissues.

Both theoretically and experimentally, PHFMs outperform the moments in reconstructing images and recognizing rotationally invariant objects [26]. Consideration of noise and other attacks confirms this. We generate a collection of rotation-invariant features by deriving a collection of two-dimensional transforms from a collection of orthogonal projection bases. We refer to these features as PHTs, which incorporate the orthogonality and invariance benefits of Zernike and pseudo-Zernike moments, without the inherent limitations of these moments [27, 28]. If you want to get a better-reconstructed picture of polarimetric imaging in bad lighting or with some things blocked out, the authors of this paper suggest using a denoising convolutional neural network model along with 3D integral imaging. We developed this model to enhance the reliability of the reconstructed image [29].

When compared to methods that are based on Cartesian grids, the research presents a uniformly sampled polar or cylindrical grid approach for picture reconstruction. This approach improves computing performance and reduces the amount of memory that is required [30]. The purpose of this research is to offer a method for reconstructing laser polarization images that makes use of wavelet transformation and deep learning. This method enhances target identification and image quality through feature extraction and edge recognition. When compared to all the rotation-invariant feature extraction methods that are currently available, PHTs are the most efficient. This article introduces several innovative computation methodologies that expedite the computation of these transforms [31].

Using the PCET, the researchers discovered that increasing the number of moments used for image reconstruction does not necessarily result in an improvement in the quality of the image that is brought back to life through the application of image moments approaches [32]. The PHTs are the most efficient rotation-invariant feature extraction approach currently available. This article discusses several innovative computation methodologies for quickly computing these transforms [33]. This work presents a technique for phase retrieval. We use the algorithm when we encounter two related sets of Fourier-transform magnitude data [34]. We consider these sets of data to originate from a single object, observed in two different polarizations through a distortion-causing medium. The bottleneck distance is used to conduct an analysis of the quality of the reconstructed images that were created utilizing the image moments. In terms of topological analysis, the bottle distance increases as the number of moments increases; however, after achieving a minimum, there was no further reduction detected. This indicates that to achieve quality in image reconstruction, the optimal number of moments is required, rather than a high number of moments. Therefore, in practical situations only a finite number of orders and repetitions are used for finding the polar function back from the image moments.

3. Research Methodology

The PHT is highly useful because of its ability to offer a resilient and concise depiction of image characteristics, specifically for jobs involving pattern identification and image analysis. The PHT is a method of describing images using the polar coordinate system. This allows PHT to capture properties that are invariant to rotation and scale, making it particularly useful for applications where objects may appear in various orientations or sizes. The harmonic basis functions used in PHT ensure superior discriminatory capability and noise resilience. Moreover, the capability of PHT to decrease dimensionality while maintaining crucial information enhances its effectiveness in many computer vision applications.

3.1. PCET

PCET is a member of a family known as PHT. PCET C_{uv} of order u in the set Z and repetition v in the set Z, where Z is the set of integers, of a function $f(r, \varphi)$ in polar domain is a complex function defined over the unit disk $|z| \leq 1$ as,

$$C_{uv} = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \overline{H_{uv}(r, \phi)}\, g(r, \phi)\, r\, dr\, d\phi \tag{1}$$

Where:

$$H_{uv}(r, \phi) = \exp\left\{i\left(2\pi u r^2 + v\phi\right)\right\}$$

The functions $H_{uv}(r, \phi)$ form an orthogonal system.

The functions $g(r, \phi)$ can be recovered from the moments by using the following relation.

$$g(r, \phi) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} C_{uv} H_{uv}(r, \phi) \tag{2}$$

The above equation represents a theoretical perspective. In practice, inverting using large number of moments is not useful, as their computation can be time and resource consuming. Therefore, in practical situations only a finite number of orders and repetitions are used for finding the polar function back from the image moments. Following equation represents this discussion by limiting the limits of u and v.

$$g(r, \phi) = \sum_{u=\min u}^{\max u} \sum_{v=\min v}^{\max v} C_{uv} H_{uv}(r, \phi) \tag{3}$$

3.2. PHT approximation

Let I be a square image of size $N \times N$. We normalize the position of the pixels by taking center of the image as origin and the column from left to right is assigned with the x-coordinate between −1 and 1. Similarly, the rows from top to bottom are assigned with y-coordinates between −1 and 1. The x-coordinate of the center of each pixel is computed by adding and dividing the x-coordinated of the left and right borders of the pixels by 2. Similarly, the y-coordinate of the center of each pixel is computed by adding and dividing the y-coordinate of the top and bottom borders of the pixels by 2. The moments are then approximated by multiplying each pixel intensity with $H_{uv}(r, \phi)$ and the area of each pixel.

Note here that (r, ϕ) are transformed from the Cartesian coordinates of the image pixel. Further, note that the computations are limited to those pixels which have $r \leq 1$.

In a similar way, by computing $H_{uv}(r, \phi)$ for a given pixel with position transformed in polar domain, corresponding intensities values can be reconstructed by multiplying the $H_{uv}(r, \phi)$ with image moments and summing them. The entire process of computing image moments and image reconstruction can be computationally challenging. We used a first-order approximation to compute the image moments. This computation is not computationally demanding and has demonstrated remarkable outcomes in the field of image processing.

3.3. Topological data analysis

The topological data analysis presents an interesting analysis of data. It helps with measuring occurrences and disappearances of features from the images. Here, the image dataset is first transformed into an abstract object called a simplicial complex. Simplicial complexes are not ordinary graphs. In a typical graph, a node can be connected to several or no nodes, meaning we only find nodes (0-simplices) and edges (1-simplices). In contrast, simplicial complexes also have higher-dimensional simplices. Using k +1 nodes, we can create a k-simplex. Various criteria give rise to difference simplicial complexes. By changing the criteria smoothly, we see a consistent variation in the simplicial complexes. This consistency variation provides appearance and disappearance of features with variation in criteria. The appearance and disappearance time in terms of criteria are independent of each other and when they are represented by showing as coordinates in two-dimensional orthogonal system. The resulting diagram is persistence diagram.

Here, we computed persistence diagrams of the original images and reconstructed images. The persistence diagrams are computed from one-dimensional and second-dimensional complexes. The change in the topological properties is then quantified in terms of change in the persistence diagram. The difference in persistence diagrams can be summarized in terms of bottleneck distance. To compute this distance, the points are mapped one to one from one persistence diagram to another persistence diagram in such a way that sum of distances is minimum, mathematically,

$$d = \inf_{\text{all mapping}} \max \| p_1 - p_2 \|_{\infty} \tag{4}$$

To compute the distances, an infinite norm was used. In the simplicial complexes in which the image forms, a point on the persistence diagram indicates the emergence and extinction of a specific feature. The bottleneck distance computes a metric using points from the persistence diagram. We pair the points on the two-persistence diagram so that even a small alteration in the image results in the two-persistence diagrams appearing the same.

In persistent homology, bottleneck distances measure how similar two-persistence diagrams are by highlighting the most important differences between traits that are similar. At various points in time, the behavior of bottleneck distances corresponds to changes in the topological characteristics of the underlying data. Significant topological events, such as the formation or elimination of large-scale structures, can cause a dramatic increase in the bottleneck distance, suggesting significant changes. In contrast, when there is stability, the bottleneck distance does not change significantly, indicating that topological characteristics stay consistent throughout time. Because they can find changes in the topology, bottleneck distances are useful for looking at how complex data structures change over time and how long they last.

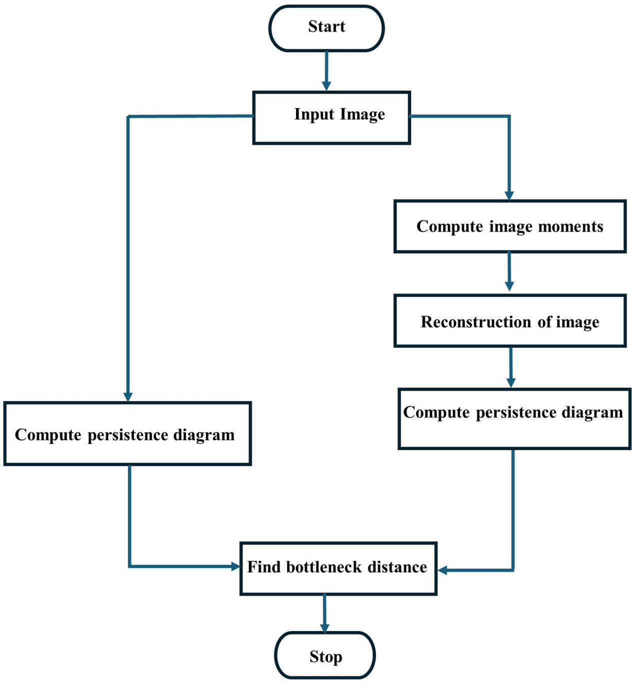
3.4. Dataset and algorithms

For the applications of topological data analysis, we selected standard images used in image processing task as shown in Figure 1. The following algorithm is used to analysis the variation in image reconstruction. The flowchart of the algorithm is shown in Figure 2. The main components of the algorithm are listed below:

1) Find image moments for an image.
2) Use image reconstruction to approximate the original image.

Figure 1
Test images used for the analysis of topological changes in image reconstruction

Figure 2
Flowchart of the algorithm



3) Find persistence diagrams of first and second order for the original image.
4) Find persistence diagrams of first and second order for the reconstructed image.
5) Find the bottleneck distances of both the orders separately.
6) Form a distribution of the bottleneck distances.

4. Experimental and Result Analysis

In Figure 3, we show the reconstructed images from the standard images shown in Figure 1. The quality of reconstructed image depends on the number of moments. The maximum number of u and v varied between 5 and 100 at an interval of 5. The visual analysis shows that an ideal number of u and v must be around 30, when zeroth order approximation was used for integration involved in the image reconstruction. Our analysis includes the study of topological changes in the image after reconstruction as compared to original image. Earlier in this article we show how to find the difference between two images topologically by analyzing the bottleneck distance between the persistence diagrams. The manuscript also studies the variation on topological changes in terms of bottleneck distances with variations in number of moments used for the image reconstruction. In Figure 4, the variation in bottleneck distances for the persistence diagram of first-dimensional homology, with respect to variation in number of moments, is shown. The number of moments taken is between 5 and 100 with an interval of 5. The image shows that the bottleneck distance decreases when we increase the number of moments. However, after finding some optimal moments, the bottleneck distance starts increasing. Figure 3

Similarly, in Figure 5, the variation in bottleneck distances for the persistence diagram of second-dimensional homology, with respect to variation in number of moments, is shown. The number of moments taken is between 5 and 100 with an interval of 5. The image shows that the bottleneck distance decreases when we

Figure 3
Reconstructed images from the test images used for the analysis of topological changes in image reconstruction

Figure 4
Bottleneck distance with respect to moments of the difference between persistence diagram of 12 test images created from the persistence diagram of first-dimensional homology

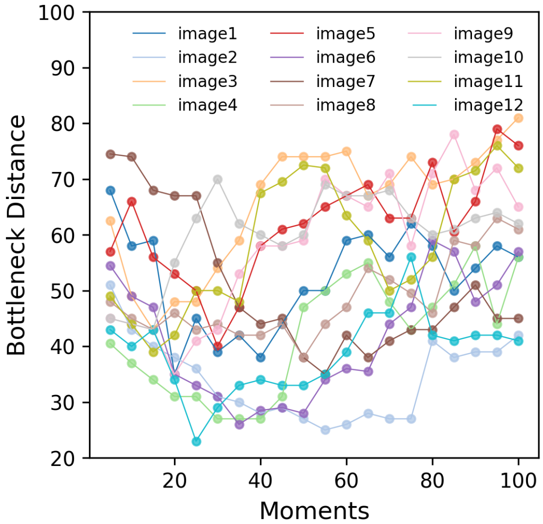
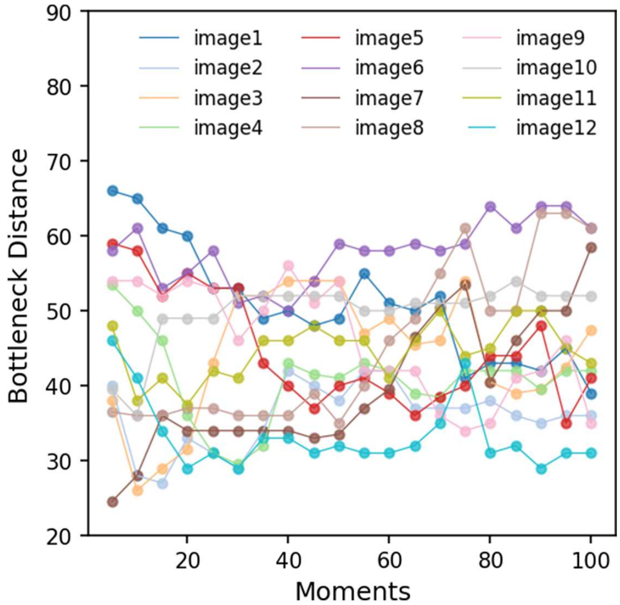


Figure 5
Bottleneck distance with respect to moments of the difference between persistence diagram of 12 test images created from the persistence diagram of first-dimensional homology



increase the number of moments. However, after finding some optimal moments, the bottleneck distance starts increasing. However, the increase in bottleneck distance is not as prominent as in first-dimensional homology.

In Table 1, bottleneck distance with respect to moments of the difference between persistence diagram of 12 test images created from the persistence diagram of first and second-dimensional homology is shown. The bottleneck distance is averaged over the 12 images to show the conclusion. Table 1 also shows that bottleneck distances are showing a decreasing and then increasing pattern with increase in number of moments for the construction of images. However, the pattern is more apparent in one-dimensional homology. In second-dimensional homology, the bottleneck distance becomes more like constant after increasing some moments.

Table 1
Bottleneck distance with respect to moments of the difference between persistence diagram of 12 test images created from the persistence diagram of first- and second-dimensional homology. The bottleneck distance is averaged over the 12 images to show the conclusion

Moments	1st dimensional	2nd dimensional
10	49.42	43.42
20	43.25	42.58
30	42.75	42.13
40	46.17	44.67
50	49.04	44.46
60	53.45	44.21
70	52.92	45.63
80	55.42	43.75
90	57.63	46.17
100	59.5	45.58

5. Conclusion

This article analyzes the topological changes that occur during image reconstruction using the image moments. We compute the topological changes by comparing the bottleneck distances between the original and reconstructed persistence diagrams. The persistence diagrams are computed from one-dimensional and second-dimensional complexes. The change in the topological properties is then quantified in terms of change in the persistence diagram. The difference in persistence diagrams can be summarized in terms of bottleneck distance. We reconstructed the images from standard test images used in image processing tasks. The image moments are computed using the PCET. This transformation is invertible with high accuracy provided enough number of moments are taken for the inversion. The transformations are not computationally intensive; therefore, the method can be applied for large-scale and real-time applications. However, in practice inverting using large number of moments is not useful, as their computation can be time and resource consuming. The number of moments between 5 and 100 at an interval of 5 is used for image reconstruction. The bottleneck distance analysis between the original and the reconstructed images shows that using large number of moments do not improve the results. An optimal number of moments are required to generate the image back. Utilizing enough moments in the process of image reconstruction presents a robust method for enhancing the quality and efficiency of the procedure, especially in applications that include large-scale and real-time operations. Moments can capture the fundamental characteristics of an image, decrease unwanted disturbances and imperfections, enhance the effectiveness of computational processes, improve the stability and convergence of algorithms, enable efficient real-time processing, and seamlessly integrate with machine learning methods. Moments are critical to achieving high-quality and efficient image reconstructions. However, the computational approach may also change the topology of the reconstructed images. The accuracy of the computed image moments can also alter the variations in topological analysis of the image reconstruction. The computation approach and their variation in topological differences between the original and reconstructed image could be an interesting analysis, which we will try to pursue in the future. Further, the integration involved in the image moment computation is simplified using zeroth order approximation. The accurate moment computation can be achieved using the actual computation of the integration. Further research can focus on finding alternative methods to find accurate image moments.

Recommendations

The finding revealed that the increase in moments for image reconstruction will not necessarily increase the quality of the image reconstructed using the image moment techniques, using PCET. The quality of reconstructed images using the image moments is analyzed based on bottleneck distance. The bottle distance increases with an increasing number of moments but after reaching a minimum further reduction was not observed indicating that to get quality in image reconstruction, an optimal number of moments is required not the large number of moments as far as topological analysis is concerned.

Acknowledgement

The authors are grateful to Gudhi and Cripser teams for creating and managing the libraries which are used for the formation of persistent diagrams and computation of the bottleneck distances.

Ethical Statement

This study does not contain any studies with human or animal subjects performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Author Contribution Statement

Manoj Kumar Singh: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing – original draft, Visualization, Project administration. **Deepika Saini:** Conceptualization, Methodology, Validation, Formal analysis, Investigation, Resources, Data curation, Writing – original draft, Writing – review & editing, Visualization. **Sanoj Kumar:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Resources, Writing – original draft, Writing – review & editing, Visualization, Supervision, Project administration.

References

[1] Aberna, P., & Agilandeeswari, L. (2024). Digital image and video watermarking: Methodologies, attacks, applications, and future directions. *Multimedia Tools and Applications, 83*(2), 5531–5591. https://doi.org/10.1007/s11042-023-15806-y

[2] Liu, X., Wu, Y., Shao, Z., & Wu, J. (2020). The modified generic polar harmonic transforms for image representation. *Pattern Analysis and Applications, 23*(2), 785–795. https://doi.org/10.1007/s10044-019-00840-0

[3] Tiwari, A., & Srivastava, V. K. (2024). Image watermarking techniques based on Schur decomposition and various image invariant moments: A review. *Multimedia Tools and Applications, 83*(6), 16447–16483. https://doi.org/10.1007/s11042-023-16109-y

[4] Singh, M. K., Kumar, S., Ali, M., & Saini, D. (2021). Application of a novel image moment computation in X-ray and MRI image watermarking. *IET Image Processing, 15*(3), 666–682. https://doi.org/10.1049/ipr2.12052

[5] Tang, Y., Li, K., Wang, C., Bian, S., & Huang, Q. (2024). A two-stage robust reversible watermarking using polar harmonic transform for high robustness and capacity. *Information Sciences, 654,* 119786. https://doi.org/10.1016/j.ins.2023.119786

[6] Hosny, K. M., & Darwish, M. M. (2019). A kernel-based method for fast and accurate computation of PHT in polar coordinates. *Journal of Real-Time Image Processing, 16,* 1235–1247. https://doi.org/10.1007/s11554-016-0622-y

[7] Tang, Y., Wang, C., Xiang, S., & Cheung, Y. M. (2024). A robust reversible watermarking scheme using attack-simulation-based adaptive normalization and embedding. *IEEE Transactions on Information Forensics and Security, 19,* 4114–4129. https://doi.org/10.1109/TIFS.2024.3372811

[8] Ma, B., Chang, L., Wang, C., Li, J., Wang, X., & Shi, Y. Q. (2020). Robust image watermarking using invariant accurate polar harmonic Fourier moments and chaotic mapping. *Signal Processing, 172,* 107544. https://doi.org/10.1016/j.sigpro.2020.107544

[9] Hosny, K. M., Darwish, M. M., & Aboelenen, T. (2020). Novel fractional-order polar harmonic transforms for grayscale and color image analysis. *Journal of the Franklin Institute, 357*(4), 2533–2560. https://doi.org/10.1016/j.jfranklin.2020.01.025

[10] Qi, M., Li, B. Z., & Sun, H. (2015). Image watermarking via fractional polar harmonic transforms. *Journal of Electronic Imaging, 24*(1), 013004. https://doi.org/10.1117/1.JEI.24.1.013004

[11] Jing, W., Ang, L. W., Palaniappan, S., & He, B. (2023). Robust image watermarking with quaternion fractional-order polar harmonic-Fourier moments based on wavelet transformation: Resistance against rotation attacks. *Journal of Informatics and Web Engineering, 2*(1), 56–75. https://doi.org/10.33093/jiwe.2023.2.1.6

[12] Yap, P. T., Jiang, X., & Kot, A. C. (2010). Two-dimensional polar harmonic transforms for invariant image representation. *IEEE Transactions on Pattern Analysis and Machine Intelligence, 32*(7), 1259–1270. https://doi.org/10.1109/TPAMI.2009.119

[13] Ali, M., Kumar, S., Pal, R., Singh, M. K., & Saini, D. (2023). Graph- and machine-learning-based texture classification. *Electronics, 12*(22), 4626. https://doi.org/10.3390/electronics12224626

[14] Pal, R., Kumar, S., & Singh, M. K. (2024). Topological data analysis and image visibility graph for texture classification. *International Journal of System Assurance Engineering and Management.* Advance online publication. https://doi.org/10.1007/s13198-024-02272-4

[15] Qi, S., Zhang, Y., Wang, C., Zhou, J., & Cao, X. (2023). A survey of orthogonal moments for image representation: Theory, implementation, and evaluation. *ACM Computing Surveys, 55*(1), 1–35. https://doi.org/10.1145/3479428

[16] Qu, Z. (2006). Algebraic reconstruction technique in image reconstruction based on data mining. *International Journal of Data Warehousing and Mining, 2*(3), 1–15. https://doi.org/10.4018/jdwm.2006070101

[17] Yang, S., & Deng, A. (2024). Quaternion fast and accurate polar harmonic Fourier moments for color image analysis and object recognition. *Journal of the Optical Society of America A, 41*(5), 852–862. https://doi.org/10.1364/JOSAA.514567

[18] Zhang, Q., Wang, C., Ma, B., Xia, Z., Li, J., Zhang, H., & Li, Q. (2023). Sedenion polar harmonic Fourier moments and their application in multi-view color image watermarking. *Signal Processing, 209,* 109010. https://doi.org/10.1016/j.sigpro.2023.109010

[19] Chen, J., Luo, Z., Zhang, Z., Huang, F., Ye, Z., Takiguchi, T., & Hancock, E. R. (2019). Polar transformation on image features for orientation-invariant representations. *IEEE Transactions on Multimedia, 21*(2), 300–313. https://doi.org/10.1109/TMM.2018.2856121

[20] Hu, H., Yang, S., Li, X., Cheng, Z., Liu, T., & Zhai, J. (2023). Polarized image super-resolution via a deep convolutional neural network. *Optics Express, 31*(5), 8535–8547. https://doi.org/10.1364/OE.479700

[21] Wang, Y., Escuti, M. J., & Kudenov, M. W. (2019). Snapshot channeled imaging spectrometer using geometric phase holograms. *Optics Express, 27*(11), 15444–15455. https://doi.org/10.1364/OE.27.015444

[22] Lv, X., Yang, Z., Wang, Y., Zhou, K., Lin, J., & Jin, P. (2021). Channeled imaging spectropolarimeter reconstruction by neural networks. *Optics Express, 29*(22), 35556–35569. https://doi.org/10.1364/OE.441850

[23] Shen, H., Lin, L., Li, J., Yuan, Q., & Zhao, L. (2020). A residual convolutional neural network for polarimetric SAR image super-resolution. *ISPRS Journal of Photogrammetry and Remote Sensing, 161,* 90–108. https://doi.org/10.1016/j.isprsjprs.2020.01.006

[24] Wu, Q., Gao, K., Li, M., Zhang, Z., Hua, Z., Zhao, H., ..., & Yu, P. (2021). Image reconstruction using variable exponential function regularization for wide-field polarization modulation imaging. *IEEE Access, 9,* 55606–55629. https://doi.org/10.1109/ACCESS.2021.3071760

[25] Terry, S. K., Lu, J. R., Turri, P., Ciurlo, A., Gautam, A. K., Do, T., ..., & Witzel, G. (2023). AIROPA IV: Validating point spread function reconstruction on various science cases. *Journal of Astronomical Telescopes, Instruments, and Systems, 9*(1), 018003. https://doi.org/10.1117/1.JATIS.9.1.018003

[26] Wang, C., Wang, X., Xia, Z., Ma, B., & Shi, Y. Q. (2020). Image description with polar harmonic Fourier moments. *IEEE Transactions on Circuits and Systems for Video Technology, 30*(12), 4440–4452. https://doi.org/10.1109/TCSVT.2019.2960507

[27] Yang, Z., Ahrary, A., & Kamata, S. I. (2010). Fast polar harmonic transforms. *The Journal of the Institute of Image Electronics Engineers of Japan, 39*(4), 399–408. https://doi.org/10.11371/iieej.39.399

[28] Kim, H. S., & Lee, H. K. (2003). Invariant image watermark using Zernike moments. *IEEE Transactions on Circuits and Systems for Video Technology, 13*(8), 766–775. https://doi.org/10.1109/TCSVT.2003.815955

[29] Usmani, K., O'Connor, T., & Javidi, B. (2021). Three-dimensional polarimetric image restoration in low light with deep residual learning and integral imaging. *Optics Express, 29*(18), 29505–29517. https://doi.org/10.1364/OE.435900

[30] Chaudhary, S. K., Wahi, P., & Munshi, P. (2023). Uniformly sampled polar and cylindrical grid approach for 2D, 3D image reconstruction using algebraic algorithm. *NDT & E International, 140,* 102960. https://doi.org/10.1016/j.ndteint.2023.102960

[31] Thibaudeau, C., Leroux, J. D., Fontaine, R., & Lecomte, R. (2013). Fully 3D iterative CT reconstruction using polar coordinates. *Medical Physics, 40*(11), 111904. https://doi.org/10.1118/1.4822514

[32] Yang, H. Y., Qi, S. R., Niu, P. P., & Wang, X. Y. (2020). Color image zero-watermarking based on fast quaternion generic polar complex exponential transform. *Signal Processing: Image Communication, 82,* 115747. https://doi.org/10.1016/j.image.2019.115747

[33] Hoang, T. V., & Tabbone, S. (2014). Fast generic polar harmonic transforms. *IEEE Transactions on Image Processing, 23*(7), 2961–2971. https://doi.org/10.1109/TIP.2014.2322933

[34] Hunt, B. R., Overman, T. L., & Gough, P. (1998). Image reconstruction from pairs of Fourier-transform magnitude. *Optics Letters, 23*(14), 1123–1125. https://doi.org/10.1364/OL.23.001123

How to Cite: Singh, M. K., Saini, D., & Kumar, S. (2024). Topological Analysis of Image Reconstruction Based on Polar Complex Exponential Transform. *Journal of Computational and Cognitive Engineering*. https://doi.org/10.47852/bonviewJCCE42023298