Received: 4 April 2022 | Revised: 16 May 2022 | Accepted: 25 May 2022 | Published online: 27 May 2022

RESEARCH ARTICLE

Journal of Computational and Cognitive Engineering
2022, Vol. 00(00) 1–13
DOI: 10.47852/bonviewJCCE2202206

BON VIEW PUBLISHING

Methods for Multiple Attribute Group Decision Making Based on Picture Fuzzy Dombi Hamy Mean Operator

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Abstract: In this work, ambiguity and unclarity are coped with the effective tools of picture fuzzy sets (PFSs), especially where the conditions demand simulation of various dimensions for evaluation, for example, decision making. PFS requires operators to measure the coordination of two PFSs. As far as this article is concerned, we bring new operators to PFSs with an application, validating this as the generalization of the concept of fuzzy sets and intuitionistic fuzzy sets. The hybrid structure of PFSs has been incorporated with other operators to develop picture fuzzy Dombi Hamy mean operator, picture fuzzy weighted Dombi Hamy mean operator, picture fuzzy Dombi dual Hamy mean operator, and picture fuzzy weighted Dombi dual Hamy mean operator. Further, the properties such as idempotency, monotonicity, boundedness, and commutativity related to each proposed operator have been discussed. By using these operators, the multiple attribute group decision-making methods are proposed. Moreover, we have explained the application by providing an example of a car supplier. The results are concluded by selecting the best car on the basis of attributes such as quality, production, service efficiency, and risk factors using operators defined on PFSs. A comparative study is also conducted to study the significance of the developed work.

Keywords: picture fuzzy sets, PFDHM operator, PFWDHM operator, PFDDHM operator, PFWDDHM operator, MAGDM

1. Introduction

In various parts of life, in order to cope with several issues like machine learning, multi-attribute decision making (MADM), and multiple attribute group decision making (MAGDM), it is necessary to compare things. MAGDM is the important for the deciding science whose objective is to get the best choice from a group of similar choices. Originally, MAGDM needs to evaluate the alternate options by many other categories, for example, single, span, and like for the objective of the evaluation. But as may be, in various other conditions it is commonly the effort of leading for MAGDM in a new manner. To deal with the above-said problems, there are many ways, but when the data are in fuzzy form, the operators are found outstanding. The under consideration article for the most part is related to the picture fuzzy (PF) operators as it is the generalized production of the fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs). Therefore, it is appropriate to mention the pioneers and recent studies in terms of the development and applications of FS and IFS.

There is plenty of unclear, ambiguous, and unstable data in real life. To handle such a situation, Zadeh (1965) introduced FS, in which each element of uncertainty is assigned with a membership grade (MG) denoted as t. Just the MG is taken in the FSs and a one-minus gradation is taken as a non-membership grade (NMG). So, it is sure to find the NMG by taking into consideration the MG. Nonetheless, in practical life, one is not sure regarding the NMG owing to the knowledge of MG. In such circumstances, it is recommended that there should be a free NMG function. To cope with the situation, Atanassov (1986) developed the idea of IFS, in which each element is assigned with MG as well as NMG denoted as f, with the condition $0 \leq t + f \leq 1$. The IF and interval-valued parameterized soft set theory and its decision have been studied in Deli and Çağman (2015) and Deli and Karataş (2016). Some similarity measures for IFSs are developed in Deli and Çağman (2016). IFS is widely used in MADM Das et al., (2016); Lin et al., (2007); Ye, (2009) and (2010); Liu, & Wang, (2007); Xu, (2007).

An IF has not been able to deliver in some cases. For example, if a person is given $t = 0.7$ and $f = 0.5$, in such condition, IFS will be unable to manage the situation, that is, $t + f = 0.7 + 0.5 = 1.2 \notin [0, 1]$. In such situation, IFS has not been kept in mind. In the same way, some problems were faced in real-life matters where the IFS has also deviated. Because of these limitations of the IFS, Yager (2013) and Yager and Abbasov (2013) initiated the system of Pythagorean FSs, containing both functions t and f with the condition $0 \leq t^2 + f^2 \leq 1$, extending the space of IFSs. In IF theory, lots of contributions have been done. Keeping that in mind nowadays, MADM plays a vital role in decision theory. Intuitionistic fuzzy entropy is developed in Burillo and Bustince (2001). For MAGDM, the intuitionistic preference relations are developed by Xu (2007). Some induced correlated aggregating operators with intuitionistic fuzzy numbers (IFNs) proposed in Wei and Zhao (2012). Some Einstein hybrid aggregation operators are developed in Zhao and Wei (2013). The generalized

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interactive geometric interaction operators are developed by Garg (2016). With IFNs, the model of MAGDM that considers the additive consistency and group consensus at a time is done in Chu et al., (2016). Dombi T-norm (DTN) and T-conorm (DTCN) operations are developed by Dombi (1982). Further, the Dombi operations in the context of IFS are developed in Liu, et al., (2018); Chen and Ye, (2017); Li, et al., (2018).

IFSs effectively improved FSs; however, in such a situation when there are in excess of two free circumstances like in casting a ballot (choice of inclusion, restraint, resistance, and exclusion), IFSs neglected to depict the circumstance. Understanding this, Cuong (2013 and 2014) built up a new direction known as picture fuzzy set (PFS), which described the MG, refusal grade (RG) denoted as h, and NMG of an element or object in interval $[0, 1]$, with the condition $0 \leq t + h + f \leq 1$. PFSs extend the model of FSs and IFSs. As research hotspots of PFSs, operators are used to getting the best choice from a group of similar choices. Some PF operators are discussed in Liu and Zhang (2018) and Cuong and Pham (2015). MADM problems, based on bipolar valued PF operators, are discussed by Riaz et al., (2018). PF Dombi aggregation operators are discussed in Jana et al., (2019). Many problems related to MADM based on PF aggregation operators, Dombi Heronian mean operators, PF Hamacher operators, complex PF Hamacher operators, and Bonferroni mean operators are discussed in Garg (2017) and Zhang et al., (2018). Due to a lot of research in PF environment, and by investigating the drawbacks of IF environment we aim to develop the novel operators in PF environment as it is the generalized structure of the fuzzy sets with three membership functions. The main advantage of the work we have done in this article is that it can be used to solve the problems given in PF context and also we have examined that the result obtained by applying the proposed work is exactly the same as many other existing structures give. On the other hand, the existing structures in IF environment are enabled to solve the problems that are in PF context. To overcome the above limitations of existing operators, we have introduced novel operators with some features in this article.

Keeping in view the limitations of IFS for example in a situation when there are in excess of two free circumstances like in casting a ballot (choice of inclusion, restraint, resistance, and exclusion), the IFS fails to handle the situation. To handle the situation, we have extended the domain of the HM operator, DHM operator, and the DDHM operator in the context of PF information. So, the operators, intuitionistic fuzzy DHM (IFDHM) operator, intuitionistic fuzzy DDHM (IFDDHM) operator, and intuitionistic fuzzy WDDHM (IFWDDHM), developed in Cuong (2014) are extended in the context of PF information. Following this, the purpose of this article is to

1. Develop the hybrid structure of PFSs and HM operator, DHM operator, DDHM operator.
2. Investigate the properties such as idempotency, monotonicity, boundedness, and commutativity of the proposed operators.
3. Develop new models to solve the MAGDM problems related to the proposed operators.
4. Develop a numerical example of a car supplier. In a nutshell, we concluded our results by selecting the best car on the basis of attributes such as quality, production, service efficiency, and risk factor

The proposed article is organized as follows. The basic work that is necessary to study the proposed article is given in the next section. In Section 3, PFHM operators based on DTN and DTCN are proposed. Section 4 investigates the application of proposed work on MAGDM problems in the PF context. Section 5 refers to the numerical example of MAGDM, and a comparative study is made by comparing the data given in Liu and Zhang (2018). In the last section, we have concluded the article.

2. Preliminaries

The ideas studied in this section provided the basis for proposed operators; X acts as universal set.

Definition 1: An *IFS* is of the form $k = \{(t(x),\ f(x))\ such\ that\ x \in X\}$ where t and f are functions from X to an element in unit interval $[0, 1]$ with a restriction $0 \leq t + f \leq 1$ and $r = 1 - (t + f)$ is the hesitant grade (HG) of x in k, where (t, f) is considered as an IFN (Atanassov, 1986).

Now, we will define the concept of IF Dombi mean operators based on DTN and DTCN.

Definition 2: For IFNs $k_i = (t_{i_j}, f_{i_j})(i = 1, 2, \ldots, n)$, the IFDHM operator can be defined as: $IFDHM^{(x)}(k_i) = \frac{1}{C_n^x}\left(\oplus_{1\leq i_1 < \ldots < i_x \leq n}\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}\right)$, where x is a parameter and $x = 1, 2, \ldots n$, $i_1, i_2, \ldots, i_x$, are x integer values taken from the set $\{1, 2, \ldots, n\}$ of n integers values, and $C_n^x = \frac{n!}{x!(n-x)!}$. (Li et al, 2018).

Theorem 1: For IFNs $k_i (i = 1, 2, \ldots, n)$, Definition 2 results an IFN and has (Li et al, 2018).

$$IFDHM^{(x)}(k_i) = \frac{1}{C_n^x}\left(\oplus_{1\leq i_1<\cdots<i_x\leq n}\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}\right)$$

$$= \left(1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{1-t_{i_j}}{t_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\frac{1}{\sum_{j=1}^{x}\left(\frac{f_{i_j}}{1-f_{i_j}}\right)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right).$$

Further, we use: $T'_{i_j} = \sum_{j=1}^{x}\left(\frac{1-t_{i_j}}{t_{i_j}}\right)^{\lambda}$, $H'_{i_j} = \sum_{j=1}^{x}\left(\frac{h_{i_j}}{1-h_{i_j}}\right)^{\lambda}$ and $F'_{i_j} = \sum_{j=1}^{x}\left(\frac{f_{i_j}}{1-f_{i_j}}\right)^{\lambda}$, for θ representation $T'_{\theta_j} = \sum_{j=1}^{x}\left(\frac{1-t_{\theta_j}}{t_{\theta_j}}\right)^{\lambda}$, $H'_{\theta_j} = \sum_{j=1}^{x}\left(\frac{h_{\theta_j}}{1-h_{\theta_j}}\right)^{\lambda}$, $F'_{\theta_j} = \sum_{j=1}^{x}\left(\frac{f_{\theta_j}}{1-f_{\theta_j}}\right)^{\lambda}$.

Definition 3: For IFNs $k_i (i = 1, 2, \ldots, n)$ having the weight vector (WV) $w = (w_1, w_2, \ldots, w_n)^T$ with restriction $w_i \in [0, 1]$ and $\sum_{i=1}^{x} w_i = 1$, the IFWDHM operator can be defined as (Li et al, 2018):

$$IFWDHM^{(x)}(k_i) = \begin{cases} \frac{\oplus_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x} w_{i_j}\right)\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^x} & (1 \leq x < n) \\ \otimes_{j=1}^{x} k_{i_j}^{\frac{1-w_i}{n-1}} & (x = n) \end{cases}.$$

Theorem 2: For IFNs $k_i (i = 1, 2, \ldots, n)$ having the WV $w = (w_1, w_2, \ldots, w_n)^T$ with restriction $w_i \in [0, 1]$ and $\sum_{i=1}^{x} w_i = 1$, Definition 3 results an IFN and has (Li et al, 2018):

$$IFWDHM^{(x)}(k_i) = \frac{\oplus_{1\leq i_1<\cdots<i_x\leq n}\left(1 - \sum_{j=1}^{x} w_{i_j}\right)\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^x}$$

$$= \left(1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1 - \sum_{j=1}^{x} w_{i_j}\right)\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1 - \sum_{j=1}^{x} w_{i_j}\right)\frac{1}{F'_{i_j}}\right)^{\frac{1}{\lambda}}}\right)(1 \leq x < n).$$

or,

$$IFWDHM^{(x)}(k_i) = \otimes_{j=1}^{x} k_{i_j}^{\frac{1-w_i}{n-1}}$$

$$= \left(\frac{1}{1 + \left(T'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(F'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}\right)(x = k).$$

IFSs effectively improved FSs; however, in such a situation when there are in excess of two free circumstances like in casting a ballot (choice of inclusion, restraint, resistance, and exclusion), IFSs neglected to depict the circumstance. Understanding this, the new idea was proposed in Cuong (2013) known as PFS, which described the MG, RG, and NMG.

Definition 4: A *PFS* is of the form $k = \{(t(x), i(x), f(x))$ *sh that* $x \in X\}$ where t, i, and f are functions from X to an element in unit interval $[0, 1]$ with a restriction $0 \leq t + i + f \leq 1$ and $r = 1 - (t + i + f)$ is the HG of x in k, where (t, i, f) is considered as picture fuzzy number (PFN)) (Jana et al, 2019).

Definition 5: For two PFSs $k_1 = \{(x, t_1(x), i_1(x), f_1(x))| \; x \in X\}$ and $k_2 = \{(x, t_2(x), i_2(x), f_2(x))|x \in X\}$, some basic operations are defined as (Cuong et al, 2018):

1. $k_1 \subseteq k_2$ iff $t_1(x) \leq t_2(x)$, $i_1(x) \leq i_2(x)$, $f_1(x) \geq f_2(x)$.
2. $k_1 = k_2$ iff $k_1 \subseteq k_2$ and $k_2 \subseteq k_1$.
3. $k_1 \cup k_2 = (\max(t_1(x), t_2(x)), \min(i_1(x), i_2(x)), \min(f_1(x), f_2(x)))$.
4. $k_1 \cap k_2 = (\min(t_1(x), t_2(x)), \max(i_1(x), i_2(x)), \max(f_1(x), f_2(x)))$.
5. $k_1^c = (f_1(x), i_1(x), t_1(x))$.

3. PFHM Operators

In some situations, there exist limitations when the data are in IFNs, following this we aim to develop above-discussed operators in the PF environment. Here, we have developed the picture fuzzy Dombi Hamy mean (PFDHM) operator and picture fuzzy weighted Dombi Hamy mean (PFWDHM) operator.

Definition 6: For PFNs $k_i = (t_k, h_k, f_k)(i = 1, 2, \ldots, n)$, the score function can be defined as: $S(k) = t_k - h_k - f_k$, where $S(k) \in [0, 1]$.

Definition 7: For PFNs $k_i = (t_k, h_k, f_k)(i = 1, 2, \ldots, n)$, the accuracy function can be defined as: $S(k) = t_k + h_k + f_k$, where $S(k) \in [0, 1]$.

3.1. The PFDHM operator

Definition 8: For PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})(i = 1, 2, \ldots, n)$, the PFDHM operator can be defined as: $PFDHM^{(x)}(k_i) = \frac{1}{C_n^x}\left(\oplus_{1 \leq i_1 < \cdots < i_x \leq n}\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}\right)$, where x is a parameter and $x = 1, 2, \ldots n, i_1, i_2, \ldots, i_x$, are x integer values taken from the set $\{1, 2, \ldots, n\}$ of n integers values, and $C_n^x = \frac{n!}{x!(n-x)!}$.

Theorem 3: For PFNs $k_i(i = 1, 2, \ldots, n)$, Definition 7 results a PFN and has

$$PFDHM^{(x)}(k_i) = \frac{1}{C_n^x}\left(\oplus_{1 \leq i_1 < \cdots < i_x \leq n}\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}\right)$$

$$= \left(1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{T'_{x_ij}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{H'_{x_ij}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{F'_{x_ij}}\right)^{\frac{1}{\lambda}}}\right).$$

Further, for the sake of simplicity we use

$$T_{i_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{T'_{x_ij}}\right)^{\frac{1}{\lambda}}}, \; H_{i_j} = \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{H'_{x_ij}}\right)^{\frac{1}{\lambda}}},$$

$$F_{i_j} = \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{F'_{x_ij}}\right)^{\frac{1}{\lambda}}}.$$

For PFN $\pi_i = (t_{\theta_j}, h_{\theta_j}, f_{\theta_j})$,

$$T_{\theta_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{T'_{x\theta j}}\right)^{\frac{1}{\lambda}}}, \; H_{\theta_j} = \frac{1}{1 + \left(\frac{1}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{F'_{x\theta j}}\right)^{\frac{1}{\lambda}}},$$

$$F_{\theta_j} = \frac{1}{1 + \left(\frac{1}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{F'_{x\theta j}}\right)^{\frac{1}{\lambda}}}.$$

Proof:
1. By operational rules of PFNs, and applying Definition 7 we have

$$\otimes_{j=1}^{x} k_{i_j} = \left(\frac{1}{1 + \left(T'_{i_j}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(H'_{i_j}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(F'_{i_j}\right)^{\frac{1}{\lambda}}}\right),$$

$$(\otimes_{j=1}^{x} k_{i_j})^{\frac{1}{x}} = \left(\frac{1}{1 + \left(\frac{1}{x}T'_{i_j}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(\frac{1}{x}H'_{i_j}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(\frac{1}{x}F'_{i_j}\right)^{\frac{1}{\lambda}}}\right),$$

Moreover,

$$\oplus_{1 \leq i_1 < \cdots < i_x \leq n}\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}} = \left(\begin{array}{c} 1 - \frac{1}{1 + \left(\sum_{1 \leq i_1 < \cdots < i_x \leq n} xT'_{i_j}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1 + \left(\sum_{1 \leq i_1 < \cdots < i_x \leq n} xH'_{j}\right)^{\frac{1}{\lambda}}}, \frac{1}{1 + \left(\sum_{1 \leq i_1 < \cdots < i_x \leq n} xF'_{i_j}\right)^{\frac{1}{\lambda}}} \end{array}\right),$$

Furthermore,

$$PFDHM = \frac{1}{C_n^x}\left(\oplus_{1 \leq i_1 < \cdots < i_x \leq n}(\otimes_{j=1}^{x} k_{i_j})^{\frac{1}{x}}\right) = (T_{i_j}, H_{i_j}, F_{i_j}).$$

2. Next, we prove that Theorem 1 results a PFN.
Consider, T_{i_j}, H_{i_j} and F_{i_j}. Then we are to prove,

(i) $0 \leq T_{i_j} \leq 1$, $0 \leq H_{i_j} \leq 1$, $0 \leq F_{i_j} \leq 1$.
(ii) $0 \leq T_{i_j} + H_{i_j} + F_{i_j} \leq 1$.

(i) Since $t_{i_j} \in [0, 1]$, we can get

$$1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}} \geq 1 \Rightarrow \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \leq i_1 < \cdots < i_x \leq n}\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}}} \in [0, 1] \Rightarrow T_{i_j} \in [0, 1].$$

Therefore, $0 \leq \boldsymbol{T_{i_j}} \leq 1$. Similarly, $0 \leq \boldsymbol{H_{i_j}} \leq 1$, $0 \leq \boldsymbol{F_{i_j}} \leq 1$.

(ii) Obviously, $0 \leq Tx_{i_j} + Hx_{i_j} + Fx_{i_j} \leq 1$, then $(T_{i_j} + H_{i_j} + F_{i_j}) \leq (F_{i_j} + H_{i_j} + F_{i_j}) = 1$. So, we get $0 \leq T_{i_j} + H_{i_j} + F_{i_j} \leq 1$, it implies that Definition 7 results a PFN.

Now, we will demonstrate some related properties.

3.1.1. Properties

In this section, the properties related to the newly proposed operator (Definition 7) are given.

1. **Idempotency:** If $k_i(1,2,\ldots,n)$ and k_i are PFNs and $k_i = k = (t_{i_j}, h_{i_j}, f_{i_j})$ for all $(i = 1,2,\ldots,n)$, then we get: $PFDHM = k$

Proof: Since $k = (t_{i_j}, h_{i_j}, f_{i_j})$, once again by Definition 7, we have

$$PFDHM^{(x)}(k_i) = \frac{1}{C_n^x}\left(\oplus_{1\le i_1<\cdots<i_x\le n}(\otimes_{j=1}^x k_{i_j})^{\frac{1}{x}}\right) = (T_{i_j}, H_{i_j}, F_{i_j})$$

$$= \left(1 - \frac{1}{1+\frac{1-t_{i_j}}{t_{i_j}}}, \frac{1}{1+\frac{h_{i_j}}{1-h_{i_j}}}, \frac{1}{1+\frac{f_{i_j}}{1-f_{i_j}}}\right) = \left(t_{i_j}, h_{i_j}, f_{i_j}\right)$$

$$= \left(t_j, h_j, f_j\right) = k.$$

2. **Monotonicity:** For two sets of PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})$ and $\pi_i = (t_{\theta_j}, h_{\theta_j}, f_{\theta_j})(i = 1,2,\ldots,n)$, If $t_{i_j} \ge t_{\theta_j}, h_{i_j} \le h_{\theta_j}, f_{i_j} \le f_{\theta_j}$, for all j, then: $PFDHM^{(x)}(k_i) \ge PFDHM^{(x)}(\pi_i)$.

Proof: Since $x \ge 1,\ t_{i_j} \ge t_{\theta_j} \ge 0,\ h_{\theta_j} \ge h_{i_j} \ge 0,$ and $f_{\theta_j} \ge f_{i_j} \ge 0$, then:

$$T'_{i_j} \le T'_{\theta_j} \Rightarrow \frac{1}{T'_{i_j}} \ge \frac{1}{T'_{\theta_j}} \Rightarrow \frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{T'_{i_j}}$$

$$\ge \frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{T'_{\theta_j}} \Rightarrow T_{i_j} \ge T_j.$$

In a similar manner, we can deal with the h and f.
For $k = PFDHM^{(x)}(k_i)$ and $\pi = PFDHM^{(x)}(\pi_i)$, the $S(k)$ and $S(\pi)$ are the score values. We can imply that $S(k) \ge S(\pi)$, related to the score value of PFN. Further we investigate the below cases:

(i) $S(k) > S(\pi)$, then we can get $PFDHM^{(x)}(k_i) > PFDHM^{(x)}(\pi_1, \pi_2, \ldots, \pi_n)$.

(ii) If $S(k) = S(\pi)$, then $(T_{i_j} + H_{i_j} + F_{i_j}) = (T_{\theta_j} + H_{\theta_j} + F_{\theta_j})$. As $t_{i_j} \ge t_{\theta_j} \ge 0, h_{\theta_j} \ge h_{i_j} \ge 0, f_{\theta_j} \ge f_{i_j} \ge 0$, then we can estimate that: $T_{i_j} = T_{\theta_j}$, $H_{i_j} = {}_{\theta_j}$, and $F_{i_j} = F_{\theta_j}$. So, it follows that $H(k) = H(\pi)$, that is, $PFDHM^{(x)}(k_i) = PFDHM^{(x)}(\pi_i)$.

3. **Boundedness:** For a set of PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})$, $k^+ = (t_{max_j}, h_{max_j}, f_{max_j})$ and $k^- = (t_{min_j}, h_{min_j}, f_{min_j})(i = 1,2,\ldots,k)$, then $k^+ < PFDHM^{(x)}(k_i) < k^+$.

Proof: From Properties 1 and 2, we have: $PFDHM^{(x)}(k_i) \ge PFDHM^{(x)}(k^-, k^-, \ldots, k^-) = k^-$ and $PFDHM^{(x)}(k_i) \le PFDHM^{(x)}(k^+, k^+, \ldots, k^+) = k^+$. Then, $k^- < PFDHM^{(x)}(k_i) < k^+$.

4. **Commutativity:** For two sets of PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})$ and $\pi_i = (t_{\theta_j}, h_{\theta_j}, f_{\theta_j})(i = 1,2,\ldots,n)$, then: $PFDHM^{(x)}(k_i) = PFDHM^{(x)}(\pi_1, \pi_2, \ldots, \pi_n)$, where (π_i) is any permutation of (k_i).

Proof: Since, $(\pi_1, \pi_2, \ldots, \pi_n)$ is a permutation, then:

$$\frac{1}{C_n^x}\left(\oplus_{1\le i_1<\cdots<i_x\le n}\left(\otimes_{j=1}^x k_{i_j}\right)^{\frac{1}{x}}\right) = \frac{1}{C_n^x}\left(\oplus_{1\le i_1<\cdots<i_x\le n}\left(\otimes_{j=1}^x \pi_{i_j}\right)^{\frac{1}{x}}\right),$$

Thus, $PFDHM^{(x)}(k_i) = PFDHM^{(x)}(\pi_1, \pi_2, \ldots, \pi_n)$.

3.2. The PFWDHM operator

Attribute weight plays an important role in constructive DM and also influences the outcomes. So, it is found important to deal with the weight of attributes in gathering data. Also seen that PFDHM operator fails to cope with the issue of attribute weight. To overcome this problem, we have proposed PFWDHM operator.

Definition 9: For a group of PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})(i = 1,2,\ldots,n)$ having WV $w = (w_1, w_2, \ldots, w_n)^T$ with the restriction $\sum_{i=1}^x w_i = 1$, the PFWDHM operator can be defined as

$$PFWDHM_w^{(x)}(k_i) = \begin{cases} \frac{\oplus_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\left(\otimes_{j=1}^x k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^x} (1 \le x < n) \\ \otimes_{j=1}^x k_{i_j}^{\frac{1-w_{i_j}}{n-1}} (x = n) \end{cases}.$$

Theorem 4: For a group of PFNs$k_i = (t_{i_j}, h_{i_j}, f_{i_j})(i = 1,2,\ldots,n)$ having WV $w = (w_1, w_2, \ldots, w_n)^T$ with the restriction $\sum_{i=1}^x w_i = 1$, Definition 8 results a PFN and has

$$PFWDHM_w^{(x)}(k_i) = \frac{\oplus_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\left(\otimes_{j=1}^x k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^x}$$

$$= \left(\begin{array}{c} 1 - \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{H'_{i_j}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{F'_{i_j}}\right)^{\frac{1}{\lambda}}} \end{array}\right)(1 \le x < n).$$

or

$$PFDHM_w^{(x)}(k_i) = \otimes_{j=1}^x k_{i_j}^{\frac{1-w_i}{n-1}}$$

$$= \left(\begin{array}{c} \frac{1}{1+\left(T'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1+\left(H'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(F'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}} \end{array}\right)(x = k).$$

Further, for the sake of simplicity throughout the article we use

$$wT_{i_j} = 1 - \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}}},$$

$$wH_{i_j} = \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{H'_{i_j}}\right)^{\frac{1}{\lambda}}},$$

$$wF_{i_j} = \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{F'_{i_j}}\right)^{\frac{1}{\lambda}}}.$$

For PFN $\pi_i = (t_{\theta_j}, h_{\theta_j}, f_{\theta_j})$,

$$wT_{\theta_j} = 1 - \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{T'_{\theta_j}}\right)^{\frac{1}{\lambda}}},$$

$$wH_{\theta_j} = \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{H'_{\theta_j}}\right)^{\frac{1}{\lambda}}}$$

$$wF_{\theta_j} = \frac{1}{1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\left(1-\sum_{j=1}^x w_{i_j}\right)\frac{1}{F'_{\theta_j}}\right)^{\frac{1}{\lambda}}}.$$

Proof: (1) For $(1 \leq x < n)$, we have

$$\otimes_{j=1}^{x} k_{i_j} = \left(\frac{1}{1+\left(T'_{i_j}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(H'_{i_j}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(F'_{i_j}\right)^{\frac{1}{\lambda}}} \right)$$

$$\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}} = \left(\frac{1}{1+\left(\frac{1}{x}T'_{i_j}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{1}{x}H'_{i_j}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\frac{1}{x}F'_{i_j}\right)^{\frac{1}{\lambda}}} \right)$$

Thereafter,

$$\left(1-\sum_{j=1}^{x} w_{i_j}\right)\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}} = \begin{pmatrix} 1-\frac{1}{1+\left(\left(1-\sum_{j=1}^{x} w_{i_j}\right)xT'_{i_j}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\left(1-\sum_{j=1}^{x} w_{i_j}\right)xH'_{i_j}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\left(1-\sum_{j=1}^{x} w_{i_j}\right)xF'_{i_j}\right)^{\frac{1}{\lambda}}} \end{pmatrix}.$$

Moreover,

$$\oplus_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x} w_{i_j}\right)\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}$$

$$= \begin{pmatrix} 1-\frac{1}{1+\left(\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x} w_{i_j}\right)\frac{x}{T'_{i_j}}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x} w_{i_j}\right)\frac{1}{H'_{i_j}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x} w_{i_j}\right)\frac{1}{F'_{i_j}}\right)^{\frac{1}{\lambda}}} \end{pmatrix}.$$

Therefore,

$$\frac{1}{C_n^x}\oplus_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x} w_{i_j}\right)\left(\otimes_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}} = \left(wT_{i_j}, wH_{i_j}, wF_{i_j}\right)$$

For the second case, when $(x = n)$, we get

$$\otimes_{j=1}^{x} k_{i_j}^{\frac{1-w_i}{n-1}} = \begin{pmatrix} \frac{1}{1+\left(T'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, \\ 1-\frac{1}{1+\left(H'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(F'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix}.$$

(2) Next, for the first case, when $(1 \leq x < n)$. Consider, wT_{i_j}, wH_{i_j} and wF_{i_j}, then we have to prove,

(i) $0 \leq wT_{i_j} \leq 1,\ 0 \leq wH_{i_j} \leq 1,\ 0 \leq wF_{i_j} \leq 1$.
(ii) $0 \leq wT_{i_j} + wH_{i_j} + wF_{i_j} \leq 1$.
(i) As, $a \in [0, 1]$, we have:

$$1+\left(\frac{x}{C_n^x}\sum_{1\leq i_1<\cdots<i_x\leq n}\left(1-\sum_{j=1}^{x} w_{i_j}\right)\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}} \geq 1 \Rightarrow wT_{i_j} \in [0, 1]$$

Therefore, $0 \leq wT_{i_j} \leq 1$. Similarly, we can get for $0 \leq wH_{i_j} \leq 1$ and $0 \leq wF_{i_j} \leq 1$. Since, $0 \leq wT_{i_j} + wH_{i_j} + wF_{i_j} \leq 1$, we get the following inequality:
$wT_{i_j} + wH_{i_j} + wF_{i_j} \leq wF_{i_j} + wH_{i_j} + wF_{i_j} = 1$. So, we get $0 \leq wT_{i_j} + wH_{i_j} + wF_{i_j} \leq 1$, it implies that Definition 8 results a PFN.

For $x = n$, it is easy to find the feasibility. Therefore, Definition 8 still results a PFN. We will then evaluate some of the required features of PFWDHM operator.

Now, we will demonstrate some related properties.

3.2.1. Properties

In this section, the properties related to the newly proposed operator (Definition 8) are given.

1. **Idempotency:** If $k_i (i = 1, 2, \ldots, n)$ are equal, that is, $k_i = k = (t, h, f)$, and WV meets $w_i \in [0, 1]$ and $\sum_{i=1}^{x} w_i = 1$, then $PFWDHM = k$.

Proof: Since $k_i = k = (t, h, f)$, by Theorem 4, we have
(1) For $1 \leq< n$,

$$PFWDHM_w^{(x)}(k_i) = \left(wT_{i_j}, wH_{i_j}, wF_{i_j}\right)$$

$$= \begin{pmatrix} 1-\frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x-\sum_{1\leq i_1<\cdots<i_x\leq n}\left(\sum_{i=1}^{x} w_{i_j}\right)\right)\frac{1}{(1-tt)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x-\sum_{1\leq i_1<\cdots<i_x\leq n}\left(\sum_{i=1}^{x} w_{i_j}\right)\right)\frac{1}{(h1-h)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x-\sum_{1\leq i_1<\cdots<i_x\leq n}\left(\sum_{i=1}^{x} w_{i_j}\right)\right)\frac{1}{(f1-f)^{\lambda}}\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x-C_{n-1}^{x-1}\sum_{i=1}^{x} w_{i_j}\right)\frac{1}{(1-tt)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x-C_{n-1}^{x-1}\sum_{i=1}^{x} w_{i_j}\right)\frac{1}{(h1-h)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x-C_{n-1}^{x-1}\sum_{i=1}^{x} w_{i_j}\right)\frac{1}{(f1-f)^{\lambda}}\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

Since, $\sum_{i=1}^{k} w_i = 1$, we can get

$$PFWDHM_w^{(x)}(k_i) = \left(1-\frac{1}{1+\left(\frac{1}{(1-tt)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\frac{1}{(h1-h)^{\lambda}}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\frac{1}{(f1-f)^{\lambda}}\right)^{\frac{1}{\lambda}}}\right)$$

$$= (t, h, f) = k$$

(2) For the second case, when $= n$,

$$PFWDHM_w^{(x)}(k_i) = \begin{pmatrix} \frac{1}{1+\left(T'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, \\ 1-\frac{1}{1+\left(H'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, \\ 1-\frac{1}{1+\left(F'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{1-t}{t}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \\ 1-\frac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{h}{1-h}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \\ 1-\frac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{f}{1-f}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \end{pmatrix} = (t, h, f) = k,$$

which proves the required result.

2. **Monotonicity:** For two sets of PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})$ and $\pi_i = \left(t_{\theta_j}, h_{\theta_j}, f_{\theta_j}\right)(i = 1, 2, \ldots, n)$ having WV $w_i \in [0, 1]$ with restriction $\sum_{i=1}^{x} w_i = 1$. If $t_{i_j} \geq t_{\theta_j}, h_{i_j} \leq h_{\theta_j}, f_{i_j} \leq f_{\theta_j}$, for all j, then: $PFWDHM_w^{(x)}(k_i) \geq PFWDHM_w^{(x)}(\pi_i)$.

Proof: Since $x \geq 1,\ t_{i_j} \geq t_{\theta_j} \geq 0, 0 \leq h_{i_j} \leq h_{\theta_j}$ and $0 \leq f_{i_j} \leq f_{\theta_j}$, then:

$$T'_{i_j} \leq T'_{\theta_j} \Rightarrow \frac{1}{T'_{i_j}} \geq \frac{1}{T'_{\theta_j}} \Rightarrow \left(1 - \sum_{j=1}^{x} w_{i_j}\right) \frac{1}{T'_{i_j}} \geq \left(1 - \sum_{j=1}^{x} w_{i_j}\right) \frac{1}{T'_{\theta_j}}$$

$$\Rightarrow \left(\begin{array}{c} \frac{x}{C_{n-1}^{x}} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \left(1 - \sum_{j=1}^{x} w_{i_j}\right) \frac{1}{T'_{i_j}} \geq \\ \frac{x}{C_{n-1}^{x}} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \left(1 - \sum_{j=1}^{x} w_{i_j}\right) \frac{1}{T'_{\theta_j}} \end{array} \right) \Rightarrow \left(wT_{i_j} \geq wT_{\theta_j}\right).$$

Similarly, we can prove it for h and f.

For $k = PFWDHM_w^{(x)}(k_i)$ and $\pi = PFWDHM_w^{(x)}(\pi_i)$ the $S(k)$ and $S(\pi)$ are the score. We can imply that $S(k) \geq S(\pi)$, related to the score value of PFN. Further, we investigate the below cases:

(1) If $S(k) > S(\pi)$, then we can get: $PFWDHM_w^{(x)}(k_i) > PFWDHM_w^{(x)}(\pi_i)$

(2) If $S(k) = S(\pi)$, then:

$$(wT_{i_j} - wH_{i_j} - wF_{i_j}) = (wT_{\theta_j} - wH_{\theta_j} - wF_{\theta_j}).$$

Since, $t_{i_j} \geq t_{\theta_j} \geq 0, 0 \leq f_{i_j} \leq f_{\theta_j}$, $0 \leq f_{i_j} \leq f_{\theta_j}$ and we can deduce that: $wT_{i_j} = wT_{\theta_j}$. Similarly, for h and f. Therefore, it follows that $H(k) = H(\pi)$, the $PFWDHM_w^{(x)}(k_i) = PFWDHM_w^{(x)}(\pi_i)$. In a similar way, we can prove for $x = n$.

3. **Boundedness:** For a set of PFNs $k_i = \left(t_{i_j}, f_{i_j}\right), k^+ = \left(t_{maxi_j}, f_{maxi_j}\right)(i = 1, 2, \ldots, k)$ and $k^- = \left(t_{mini_j}, f_{mini_j}\right)$ having weight vector $w_i \in [0, 1]$ with restriction $\sum_{i=1}^{x} w_i = 1$, then: $k^- \leq PFWDHM_w^{(x)}(k_i) \leq k^+$.

Proof: From Properties 5 and 6, we have

$$PFWDHM_w^{(x)}(k_i) \geq PFWDHM_w^{(x)}(k^-, k^-, \ldots, k^-) = k^-,$$

$$PFWDHM_w^{(x)}(k_i) \leq PFWDHM_w^{(x)}(k^+, k^+, \ldots, k^+) = k^+ \Rightarrow k^- \leq PFWDHM_w^{(x)}(k_i) \leq k^+.$$

4. **Commutativity:** For two sets of PFNs $k_i = \left(t_{i_j}, h_{i_j}, f_{i_j}\right)$ and $\pi_i = \left(t_{\theta_j}, h_{\theta_j}, f_{\theta_j}\right)(i = 1, 2, \ldots, n)$ having weight vector meets $w_i \in [0, 1]$ with restriction $\sum_{i=1}^{x} w_i = 1$, then $PFWDHM_w^{(x)}(k_i) = PFWDHM_w^{(x)}(\pi_i)$, where $(\pi_1, \pi_2, \ldots, \pi_n)$ is any permutation of $(k_1, k_2, \ldots, k_n)$.

Proof: Because $(\pi_1, \pi_2, \ldots, \pi_n)$ is any permutation of $(k_1, k_2, \ldots, k_n)$, then $\otimes_{j=1}^{x} k_{i_j}^{\frac{1-\omega_i}{n-1}} = \otimes_{j=1}^{x} \pi_{i_j}^{\frac{1-\omega_i}{n-1}} (x = k)$, Thus, $PFWDHM_w^{(x)}(k_i) = PFWDHM_w^{(x)}(_i)$.

Example 1: For four PFNS $k_1 = (0.3, 0.3, 0.3)$, $k_2 = (0.2, 0.1, 0.3)$, $k_3 = (0.4, 0.3, 0.2)$, $k_4 = (0.1, 0.3, 0.3)$ with the weight vector $w = (0.1, 0.4, 0.3, 0.2)$, then we used to propose PFWDHM operator to aggregate four PFNs (suppose $x = 2$, $\lambda = 2$). Let,

$$PFWDHM_w^{(2)}(k_1, k_2, k_3, k_4) = \left(\begin{array}{c} 1 - \frac{1}{1 + \left(\frac{2}{C_{4-1}^{2}} \sum_{1 \leq i_1 < \cdots < i_2 \leq 4} \left(1 - \sum_{j=1}^{2} w_{i_j}\right) \frac{1}{T'_{i_j}} \right)^{\frac{1}{2}}}, \\ \frac{1}{1 + \left(\frac{2}{C_{4-1}^{2}} \sum_{1 \leq i_1 < \cdots < i_2 \leq 4} \left(1 - \sum_{j=1}^{2} w_{i_j}\right) \frac{1}{H'_{i_j}} \right)^{\frac{1}{2}}}, \\ \frac{1}{1 + \left(\frac{2}{C_{4-1}^{2}} \sum_{1 \leq i_1 < \cdots < i_2 \leq 4} \left(1 - \sum_{j=1}^{2} w_{i_j}\right) \frac{1}{F'_{i_j}} \right)^{\frac{1}{2}}} \end{array} \right)$$

At last we get $PFWDHM_w^{(2)}(k_1, k_2, k_3, k_4) = (0.23, 0.27, 0.28)$.

3.3. The PFDDHM operator

The dual Hamy mean (DHM) operator proposed in Cuong and (2015). We aim to develop DHM operator in PF environment.

Definition 10: The DHM operator is defined as follows (Cuong and Pham 2015), $DHM^{(x)}(k_1, k_2, \ldots, k_n) = \left(\prod_{1 \leq i_1 < \cdots < i_x \leq n} \left(\frac{\sum_{j=1}^{x} k_{i_j}}{x} \right) \right)^{\frac{1}{C_n^x}}$.

Following this, we proposed a picture fuzzy Dombi dual Hamy mean (PFDDHM) operator as follows:

$$PFDDHM^{(x)}(k_i) = \left(\otimes_{1 \leq i_1 < \cdots < i_x \leq n} \left(\frac{\oplus_{j=1}^{x} k_{i_j}}{x} \right) \right)^{\frac{1}{C_n^x}}$$

where x is a parameter and $x = 1, 2, \ldots n, i_1, i_2, \ldots, i_x$, are x integer values taken from the set $\{1, 2, \ldots, n\}$ of n integer's values, and $C_n^x = \frac{n!}{x!(n-x)!}$.

Theorem 5: For a PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})(i = 1, 2, \ldots, n)$, Definition 9 results a PFN and has

$$PFDDHM^{(x)}(k_i) = \left(\otimes_{1 \leq i_1 < \cdots < i_x \leq n} \left(\oplus_{j=1}^{x} k_{i_j} x \right) \right)^{\frac{1}{C_n^x}}$$

$$= \left(\begin{array}{c} \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{T'_{i_j}} \right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{H'_{i_j}} \right)^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{F'_{i_j}} \right)^{\frac{1}{\lambda}}} \end{array} \right).$$

Further, for the sake of simplicity throughout the article we use

$$DT_{i_j} = \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{T'_{i_j}} \right)^{\frac{1}{\lambda}}}, DH_{i_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{H'_{i_j}} \right)^{\frac{1}{\lambda}}}$$

$$DF_{i_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{F'_{i_j}} \right)^{\frac{1}{\lambda}}},$$

For PFN $\pi_i = (t_{\theta_j}, h_{\theta_j}, f_{\theta_j})$,

$$DT_{\theta_j} = \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{T'_{x\theta_j}} \right)^{\frac{1}{\lambda}}}, DH_{i_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{H'_{x\theta_j}} \right)^{\frac{1}{\lambda}}},$$

$$DF_{i_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x} \sum_{1 \leq i_1 < \cdots < i_x \leq n} \frac{1}{F'_{x\theta_j}} \right)^{\frac{1}{\lambda}}}.$$

Proof:

(1) First we prove that Theorem 5 holds, we have

$$\otimes_{j=1}^{x} k_{i_j} = \left(1-\frac{1}{1+\left(T'_{i_j}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(H'_{i_j}\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(F'_{i_j}\right)^{\frac{1}{\lambda}}}\right),$$

$$\otimes_{1\le i_1<\cdots<i_x\le n}\left(\frac{\oplus_{j=1}^{x} k_{i_j}}{x}\right) = \left(\begin{array}{c}\frac{1}{1+\left(\sum_{1\le i_1<\cdots<i_x\le n}\frac{x}{T'_{i_j}}\right)^{\frac{1}{\lambda}}}, \\ 1-\frac{1}{1+\left(\sum_{1\le i_1<\cdots<i_x\le n}\frac{x}{H'_{i_j}}\right)^{\frac{1}{\lambda}}}, 1-\frac{1}{1+\left(\sum_{1\le i_1<\cdots<i_x\le n}\frac{x}{F'_{i_j}}\right)^{\frac{1}{\lambda}}}\end{array}\right)$$

$$\Rightarrow PFDDHM^{(x)}(k_i) = \left(\otimes_{1\le i_1<\cdots<i_x\le n}\left(\frac{\oplus_{j=1}^{x} k_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}$$
$$= \left(DT_{i_j}, DH_{i_j}, DF_{i_j}\right).$$

(2) Next, we have to prove that Theorem 5 results a PFN

Consider, DT_{i_j}, DH_{i_j} and DF_{i_j}, Then the following conditions are to prove

1. $0 \le DT_{i_j} \le 1,\ 0 \le DH_{i_j} \le 1,\ 0 \le DF_{i_j} \le 1$;
2. $0 \le DT_{i_j} + DH_{i_j} + DF_{i_j} \le 1$.

1. Since $t_{i_j} \in [0,1]$, we can get

$$1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}} \ge 1 \Rightarrow DT_{i_j} \in [0,1]$$

Therefore, $0 \le DT_{i_j} \le 1$. Similarly, it can be done for $0 \le DH_{i_j} \le 1$, $0 \le DF_{i_j} \le 1$.

2. Since, $0 \le DT_{i_j} + DH_{i_j} + DF_{i_j} \le 1$, we get the following inequality:$DT_{i_j} + DH_{i_j} + DF_{i_j} \le DF_{i_j} + DH_{i_j} + DF_{i_j} = 1$. So, we get $0 \le DT_{i_j} + DH_{i_j} + DF_{i_j} \le 1$.

   For $x = n$, it is easy to find the feasibility. Therefore, Definition 9 still results a PFN. Now, we will then evaluate some of the required features of PFWDHM operator.

3.3.1. Properties

Some basic properties of PFDDHM operator are as follows.

1. **Idempotency:** If $k_i(1,2,\ldots,n)$ and k_i are PFNs and $k_i = k = (t_{i_j}, f_{i_j})$ for all $(i = 1,2,\ldots,n)$ then we get $PFDDHM = k$.

Proof: From Theorem 5, we have

$$PFDDHM^{(x)}(k_i) = \left(\otimes_{1\le i_1<\cdots<i_x\le n}\left(\frac{\oplus_{j=1}^{x} k_{i_j}}{x}\right)\right)^{\frac{1}{C_n^x}}$$

$$= \left(DT_{i_j}, DH_{i_j}, DF_{i_j}\right) = \left(\begin{array}{c}\frac{1}{1+\frac{1}{\frac{t_{i_j}}{1-t_{i_j}}}}, \\ 1-\frac{1}{1+\frac{1}{\frac{1-h_{i_j}}{h_{i_j}}}}, 1-\frac{1}{1+\frac{1}{\frac{1-f_{i_j}}{f_{i_j}}}}\end{array}\right) = \left(t_{i_j}, h_{i_j}, f_{i_j}\right) = (t,h,f) = k.$$

2. **Monotonicity:** For two sets of PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})$ and $\pi_i = \left(t_{\theta_j}, h_{\theta_j}, f_{\theta_j}\right)(i = 1,2,\ldots,n)$, if $t_{i_j} \ge t_{\theta_j}, h_{i_j} \le h_{\theta_j}$ and $f_{i_j} \le f_{\theta_j}$, for all j, then $PFDDHM^{(x)}(k_i) \ge PFDDHM^{(x)}(\pi_i)$.

Proof: Since $x \ge 1$, $t_{i_j} \ge t_{\theta_j} \ge 0$, $h_{\theta_j} \ge h_{i_j} \ge 0$ and $f_{\theta_j} \ge f_{i_j} \ge 0$, we have

$$T'_{i_j} \le T'_{x_{\theta_j}} \Rightarrow \frac{1}{T'_{i_j}} \ge \frac{1}{T'_{x_{\theta_j}}} \Rightarrow 1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}} \ge 1+\left(\frac{x}{C_n^x}\sum_{1\le i_1<\cdots<i_x\le n}\frac{1}{T'_{x_{\theta_j}}}\right)^{\frac{1}{\lambda}}$$

$$\Rightarrow DT_{i_j} \ge DT_{\theta_j}.$$

In same manner, we have $DH_{i_j} \ge DH_{\theta_j}$ and $DF_{i_j} \ge DF_{\theta_j}$.

For $k = PFDDHM^{(x)}(k_i)$ and $\pi = PFWDHM^{(x)}(\pi_i)$, the $S(k)$ and $S(\pi)$ are the score. We can imply that $S(k) \ge S(\pi)$, related to the score value of PFN. Further, we investigate the below cases:

(i) $S(k) > S(\pi)$, then we can get $PFDDHM^{(x)}(k_i) > PFDDHM^{(x)}(\pi_i)$.
(ii) If $S(k) = S(\pi)$, then

$$(DT_{i_j}, DH_{i_j}, DF_{i_j}) = (DT_{\theta_j}, DH_{\theta_j}, DF_{\theta_j}).$$

As $t_{i_j} \ge t_{\theta_j} \ge 0$, $h_{\theta_j} \ge h_{i_j} \ge 0$ and $f_{\theta_j} \ge f_{i_j} \ge 0$, so we can estimate that $DT_{i_j} = DT_{\theta_j}$, $DH_{i_j} = DH_{\theta_j}$ and $DF_{i_j} = DF_{\theta_j}$.

Therefore, it follows that: $H(k) = \left(DT_{i_j} + DH_{i_j} + DF_{i_j}\right) = \left(DT_{\theta_j} + DH_{\theta_j} + DF_{\theta_j}\right) = H(\pi)$. That is, $PFDDHM^{(x)}(k_i) = PFDDHM^{(x)}(\pi_i)$.

3. **Boundedness:** For a set of PFNs $k_i = \left(t_{i_j}, h_{i_j}, f_{i_j}\right)$, $k^+ = \left(t_{max_j}, h_{max_j} f_{max_j}\right)(i = 1,2,\ldots,k)$ and $k^- = \left(t_{mini_j}, h_{mini_j}, f_{mini_j}\right)$, then $k^- < PFDDHM^{(x)}(k_i) < k^+$.

Proof: From Properties 9 and 10, we have $PFDDHM^{(x)}(k_i) \ge PFDDHM^{(x)}(k^-, k^-, \ldots, k^-) = k^-$.

$$PFDDHM^{(x)}(k_i) \le PFDDHM^{(x)}(k^+, k^+, \ldots, k^+) = k^+.$$

Then, $k^- < PFDDHM^{(x)}(k_i) < k^+$.

4. **Commutativity:** For two sets of PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})$ and $\pi_i = (t_{\theta_j}, h_{\theta_j}, f_{\theta_j})(i = 1,2,\ldots,n)$, then: $PFDDHM^{(x)}(k_i) = PFDHM^{(x)}(\pi_i)$, where $(\pi_1, \pi_2, \ldots, \pi_n)$ is any permutation of $(k_1, k_2, \ldots, k_n)$.

Proof: Since $(\pi_1, \pi_2, \ldots, \pi_n)$ is a permutation, then

$$\left(\otimes_{1\le i_1<\cdots<i_x\le n}\left(\oplus_{j=1}^{x} k_{i_j} x\right)\right)^{\frac{1}{C_n^x}} = \left(\otimes_{1\le i_1<\cdots<i_x\le n}\left(\oplus_{j=1}^{x} \pi_{i_j} x\right)\right)^{\frac{1}{C_n^x}}$$

Thus, $PFDDHM^{(x)}(k_i) = PFDDHM^{(x)}(\pi_i)$.

Example 2: For four PFNs $k_1 = (0.3, 0.3, 0.3)$, $k_2 = (0.2, 0.1, 0.3)$, $k_3 = (0.4, 0.3, 0.2)$, and $k_4 = (0.1, 0.3, 0.3)$, we used PFDHM operator to aggregate four PFNs (suppose $x = 2$, $\lambda = 2$). Let $k = PFDDHM = \left(DT_{i_j}, DH_{i_j}, DF_{i_j}\right)$. At last, we get $PFDDHM = (0.24, 0.25, 0.27)$, for $n = 4$.

3.4. The PFWDDHM operator

Attribute weight plays an important role in constructive DM and also influences the outcomes. So, it is found important cope with the weight of attributes in gathering data. Also it is seen that PFDDHM operator fails to consider the issue of attribute weight. To overcome this problem, we have proposed picture fuzzy weighted Dombi dual Hamy mean (PFWDDHM) operator.

Definition 11: For a group of PFNs $k_i = \left(t_{i_j}, h_{i_j}, f_{i_j}\right)$ $(i = 1, 2, \ldots, n)$ having WV $w = (w_1, w_2, \ldots, w_n)^T$ with the restriction $\sum_{i=1}^{x} w_i = 1$, the PFWDDHM operator can be defined as:

$$PFWDDHM_w^{(x)}(k_i) = \begin{cases} \dfrac{\otimes_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{j=1}^{x} w_{i_j}\right)\left(\oplus_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^{x}} & (1 \le x < n) \\ \oplus_{j=1}^{x} k_{i_j}^{\frac{1-w_i}{n-1}} & (x = n) \end{cases}$$

Theorem 6: For a group of PFNs $k_i = \left(t_{i_j}, h_{i_j}, f_{i_j}\right)(i = 1, 2, \ldots, n)$ having weight vector $w = (w_1, w_2, \ldots, w_n)^T$ with the restriction $\sum_{i=1}^{x} w_i = 1$, Definition 12 results a PFN and has

$$PFWDDHM_w^{(x)}(k_i) = \frac{\otimes_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{j=1}^{x} w_{i_j}\right)\left(\oplus_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}}{C_{n-1}^{x}}$$

$$= \left(\begin{array}{c} \dfrac{1}{1+\left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1-\sum_{i=1}^{x} w_{i_j}\right)\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}}},\ 1 - \dfrac{1}{1+\left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1-\sum_{i=1}^{x} w_{i_j}\right)\frac{1}{H'_{i_j}}\right)^{\frac{1}{\lambda}}}, \\ 1 - \dfrac{1}{1+\left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1-\sum_{i=1}^{x} w_{i_j}\right)\frac{1}{F'_{i_j}}\right)^{\frac{1}{\lambda}}} \end{array} \right)(1 \le x < n).$$

Further, for the sake of simplicity throughout the article we use

$$wDT_{i_j} = \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{i=1}^{x} w_{i_j}\right)\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}}},$$

$$wDH_{i_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{i=1}^{x} w_{i_j}\right)\frac{1}{H'_{i_j}}\right)^{\frac{1}{\lambda}}},$$

$$wDF_{i_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{i=1}^{x} w_{i_j}\right)\frac{1}{F'_{i_j}}\right)^{\frac{1}{\lambda}}}.$$

For PFN $\pi_i = \left(t_{\theta_j}, h_{\theta_j}, f_{\theta_j}\right)$,

$$wDT_{\theta_j} = \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{i=1}^{x} w_{i_j}\right)\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}}},$$

$$wDH_{\theta_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{i=1}^{x} w_{i_j}\right)\frac{1}{H'_{i_j}}\right)^{\frac{1}{\lambda}}},$$

$$wDF_{\theta_j} = 1 - \frac{1}{1 + \left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{i=1}^{x} w_{i_j}\right)\frac{1}{F'_{i_j}}\right)^{\frac{1}{\lambda}}}.$$

or

$$PFWDDHM_w^{(x)}(k_i) = \oplus_{j=1}^{x} k_{i_j}^{\frac{1-w_i}{n-1}}$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1+\left(\sum_{i=1}^{x}\left(\frac{1-w_i}{n-1}\right)\left(\frac{t_{i_j}}{1-t_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \\ \dfrac{1}{1+\left(\sum_{i=1}^{x}\left(\frac{1-w_i}{n-1}\right)\left(\frac{1-h_{i_j}}{h_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}, \dfrac{1}{1+\left(\sum_{i=1}^{x}\left(\frac{1-w_i}{n-1}\right)\left(\frac{1-f_{i_j}}{f_{i_j}}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$(x = n)$$

Proof: Consider,

$$\left(1 - \sum_{j=1}^{x} w_{i_j}\right)\left(\oplus_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}$$

$$= \left(\begin{array}{c} \dfrac{1}{1+\left(\left(1-\sum_{j=1}^{x} w_{i_j}\right)\frac{x}{T'_{i_j}}\right)^{\frac{1}{\lambda}}}, \\ 1 - \dfrac{1}{1+\left(\left(1-\sum_{j=1}^{x} w_{i_j}\right)\frac{x}{H'_{i_j}}\right)^{\frac{1}{\lambda}}}, 1 - \dfrac{1}{1+\left(\left(1-\sum_{j=1}^{x} w_{i_j}\right)\frac{x}{F'_{i_j}}\right)^{\frac{1}{\lambda}}} \end{array} \right).$$

Therefore,

$$\frac{1}{C_{n-1}^{x}} \otimes_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{i=1}^{x} w_{i_j}\right)\left(\oplus_{j=1}^{x} k_{i_j}\right)^{\frac{1}{x}}$$

$$= \left(wDT_{i_j}, wDH_{i_j}, wDFi_j\right)(1 \le x < n).$$

For the second case, when$(x = n)$, we get

$$\oplus_{j=1}^{x} k_{i_j}^{\frac{1-w_i}{n-1}} = \left(\begin{array}{c} \dfrac{1}{1+\left(T'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, \\ 1 - \dfrac{1}{1+\left(T'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}}, 1 - \dfrac{1}{1+\left(T'_{i_j}\left(\frac{1-w_i}{n-1}\right)\right)^{\frac{1}{\lambda}}} \end{array} \right).$$

Next, for the first case, when $(1 \le x < n)$,

Consider, wDT_{i_j}, wDH_{i_j} and wDF_{i_j}. Then we have to prove
(I) $0 \le wDT_{i_j} \le 1$, $0 \le wDH_{i_j} \le 1$ and $0 \le wDF_{i_j} \le 1$
(II) $0 \le wDT_{i_j} + wDH_{i_j} + wDF_{i_j} \le 1$.

Proof:
(I) As $wDTx_{i_j} \in [0, 1]$, we can get

$$1 + \left(\frac{x}{C_n^x}\sum_{1 \le i_1 < \cdots < i_x \le n}\left(1 - \sum_{i=1}^{x} w_{i_j}\right)\frac{1}{T'_{i_j}}\right)^{\frac{1}{\lambda}} gt; 1 \Rightarrow wDT_{i_j} \in [0, 1].$$

Therefore, $0 \le wDT_{i_j} \le 1$. Similarly, we can get $0 \le wDH_{i_j} \le 1$, $0 \le wDF_{i_j} \le 1$.
(II) Since, $0 \le wDT_{i_j} + wDH_{i_j} + wDF_{i_j} \le 1$, we get the following inequality, $wDT_{i_j} + wDH_{i_j} + wDF_{i_j} \le wDF_{i_j} + wDH_{i_j} + wDF_{i_j} = 1$. So, we get $0 \le DT_{i_j} + DH_{i_j} + DF_{i_j} \le 1$.

For $x = n$, it is easy to find the feasibility. Therefore, Definition 10 still results a PFN. We will then evaluate some of the required features of PFWDDHM operator.

3.4.1. Properties

1. **Idempotency:** If $k_i (i = 1, 2, \ldots, n)$ are equal, that is, $k_i = k = (t, h, f)$, and WV meets $w_i \in [0, 1]$ and $\sum_{i=1}^{x} w_i = 1$ then: $PFWDDHM_w^{(x)}(k_i) = k$.

Proof: Since $k_i = k = (t, h, f)$, by Theorem 6, we have

(1) For $1 \leq x < n$.

$$PFWDDHM_w^{(x)}(k_i) = \left(wDT_{i_j}, wDH_{i_j}, wDF_{i_j}\right)$$

$$= \left(\begin{array}{c} \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x - C_{n-1}^{x-1}\sum_{i=1}^x w_{i_j}\right)\frac{1}{(t1-t)^\lambda}\right)^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x - C_{n-1}^{x-1}\sum_{i=1}^x w_{i_j}\right)\frac{1}{(1-hh)^\lambda}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x - C_{n-1}^{x-1}\sum_{i=1}^x w_{i_j}\right)\frac{1}{(1-ff)^\lambda}\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

Since $\sum_{i=1}^k w_i = 1$, we can get

$$PFWDDHM_w^{(x)}(k_i) = \left(\begin{array}{c} \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x - C_{n-1}^{x-1}\right)\frac{1}{\left(\frac{t}{1-t}\right)^\lambda}\right)^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x - C_{n-1}^{x-1}\right)\frac{1}{(1-hh)^\lambda}\right)^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1+\left(\frac{1}{C_{n-1}^x}\left(C_n^x - C_{n-1}^{x-1}\right)\frac{1}{\left(\frac{1-f}{f}\right)^\lambda}\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$= \left(\frac{1}{1+\left(\frac{1}{(1-tt)^\lambda}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(\frac{1}{(h1-h)^\lambda}\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(\frac{1}{(f1-f)^\lambda}\right)^{\frac{1}{\lambda}}} \right) = (t, h, f) = k.$$

(2) For the second case, when $x = n$,

$$PFWDDHM_w^{(x)}(k_i)$$

$$= \left(1 - \frac{1}{1+\left(\sum_{i=1}^x \left(\frac{1-w_i}{n-1}\right)\left(\frac{t_{i_j}}{1-t_{i_j}}\right)^\lambda\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\sum_{i=1}^x \left(\frac{1-w_i}{n-1}\right)\left(\frac{1-h_{i_j}}{h_{i_j}}\right)^\lambda\right)^{\frac{1}{\lambda}}}, \frac{1}{1+\left(\sum_{i=1}^x \left(\frac{1-w_i}{n-1}\right)\left(1-f_{i_j} f_{i_j}\right)^\lambda\right)^{\frac{1}{\lambda}}} \right)$$

$$= \left(\frac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{t}{1-t}\right)^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{1-h}{h}\right)^\lambda\right)^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\left(\frac{n-1}{n-1}\left(\frac{1-f}{f}\right)^\lambda\right)^{\frac{1}{\lambda}}} \right) = (t, h, f) = k, \text{Proved.}$$

2. **Monotonicity:** For two sets of PFNs $k_i = (t_{i_j}, h_{i_j}, f_{i_j})$ and $\pi_i = \left(t_{\theta_j}, h_{\theta_j}, f_{\theta_j}\right)(i = 1, 2, \ldots, n)$ having WV $w_i \in [0, 1]$ with restriction $\sum_{i=1}^x w_j = 1$, If $t_{i_j} \geq t_{\theta_j}, h_{i_j} \leq h_{\theta_j}, f_{i_j} \leq f_{\theta_j}$, for all j, then: $PFWDDHM_w^{(x)}(k_i) \geq PFWDDHM_w^{(x)}(\pi_i)$.

Proof: Since $x \geq 1$, $t_{i_j} \geq t_{\theta_j} \geq 0$, $0 \leq h_{i_j} \leq h_{\theta_j}$ and $0 \leq f_{i_j} \leq f_{\theta_j}$, then: $T'_{i_j} \geq T'_{x_{\theta_j}} \Rightarrow \frac{1}{T'_{i_j}} \leq \frac{1}{T'_{x_{\theta_j}}} \Rightarrow wDT_{i_j} \geq wDT_{\theta_j}$. Similarly, we have: $wDT_{i_j} \leq wDT_{\theta_j}$ and $wDT_{i_j} \leq wDT_{\theta_j}$.

For $k = PFWDDHM_w^{(x)}(k_i)$ and $\pi = PFWDDHM^{(x)}(\pi_i)$, the $S(k)$ and $S(\pi)$ are the score. We can imply that $S(k) \geq S(\pi)$, related to the score value of PFN. Further, we investigate the below cases:

(1) If $S(k) > S(\pi)$, then we can get $PFWDDHM_w^{(x)}(k_i) > PFWDDHM_w^{(x)}(\pi_i)$.
(2) If $S(k) = S(\pi)$, then

$$(wDT_{i_j} - wDH_{i_j} - wDF_{i_j}) = (wDT_{\theta_j} - wDH_{\theta_j} - wDF_{\theta_j}).$$

Since, $t_{i_j} \geq t_{\theta_j} \geq 0, 0 \leq h_{i_j} \leq h_{\theta_j}$ and $0 \leq f_{i_j} \leq f_{\theta_j}$, we can deduce that: $wDT_{i_j} = wDT_{\theta_j}$, $wDH_{i_j} = wDH_{\theta_j}$ and $wDF_{i_j} = DF_{\theta_j}$. Therefore, it follows that $H(k) = H(\pi)$. That is, $PFWDDHM_w^{(x)}(k_i) = PFWDDHM_w^{(x)}(\pi_i)$. Similarly, for $x = n$.

3. **Boundedness:** For a set of PFNs $k_i = \left(t_{i_j}, h_{i_j}, f_{i_j}\right)$, $k^+ = \left(t_{maxi_j}, h_{maxi_j}, f_{maxi_j}\right)(i = 1, 2, \ldots, k)$ and $k^- = \left(t_{mini_j}, h_{mini_j}, f_{mini_j}\right)$ having WV $w_i \in [0, 1]$ with the restriction $\sum_{i=1}^x w_i = 1$, then $k^- \leq PFWDDHM_w^{(x)}(k_i) \leq k^+$.

Proof: Based on Properties 13 and 14, we have $PFWDDHM_w^{(x)}(k_i) \geq PFWDDHM_w^{(x)}(k^-, k^-, \ldots, k^-) = k^-$, $PFWDDHM_w^{(x)}(k_i) \leq PFWDDHM_w^{(x)}(k^+, k^+, \ldots, k^+) = k^+$.

Then we have $k^- \leq PFWDDHM_w^{(x)}(k_i) \leq k^+$.

4. **Commutativity:** For two sets of PFNs $k_i = \left(t_{i_j}, h_{i_j}, f_{i_j}\right)$ and $\pi_i = \left(t_{\theta_j}, h_{\theta_j}, f_{\theta_j}\right)(i = 1, 2, \ldots, n)$ having weight vector meets $w_i \in [0, 1]$ with restriction $\sum_{i=1}^x w_i = 1$, then $PFWDDHM_w^{(x)}(k_i) = PFWDDHM^{(x)}(\pi_1, \pi_2, \ldots, \pi_n)$, where $(\pi_1, \pi_2, \ldots, \pi_n)$ is any permutation of $(k_1, k_2, \ldots, k_n)$.

Proof: Because $(\pi_1, \pi_2, \ldots, \pi_n)$ is a permutation of $(k_1, k_2, \ldots, k_n)$, then:
$\frac{1}{C_{n-1}^x}(\otimes_{1 \leq i_1 < \cdots < i_x \leq n}(1 - \sum_{j=1}^x w_{i_j})(\oplus_{j=1}^x k_{i_j})^{\frac{1}{x}}) =$
$\frac{1}{C_{n-1}^x}(\otimes_{1 \leq i_1 < \cdots < i_x \leq n}(1 - \sum_{j=1}^x w_{i_j})(\oplus_{j=1}^x \pi_{i_j})^{\frac{1}{x}})$, for $(1 \leq x < k)$ and $\oplus_{j=1}^x k_{i_j}^{\frac{1-w_i}{n-1}} = \oplus_{j=1}^x \pi_{i_j}^{\frac{1-w_i}{n-1}}$ for $(x = k)$. Thus, $PFWDDHM_w^{(x)}(k_i) = PFWDDHM^{(x)}(_i)$.

4. A MAGDM Approach Based on the Proposed Operators

This section refers to the application of the proposed operators to deal with the problem of MAGDM with PFNs. Suppose $X = \{x_1, x_2, \ldots, x_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$ are the sets of alternatives and attributes, respectively. The weight vector of C be $w = \{w_1, w_2, \ldots, w_n\}$ with restriction $w_j \in [0, 1]$ and $\sum_{j=1}^x w_i = 1$. We may find connoisseurs $Y = \{y_1, y_2, \ldots, y_z\}$ who are called to assess data and their weight vector is $\varpi = \{\varpi_1, \varpi_2, \ldots, \varpi_z\}^T$ with $\varpi_t \in [0, 1], (t = 1, 2, \ldots, z)$, $\sum_{t=1}^z \varpi_t = 1$. The expert y_t evaluates each attributes c_j of each alternative x_i by the form of PFN $a_{ij}^t = \left(t_{ij}^t, h_{ij}^t, f_{ij}^t\right)(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ and then the decision matrix $A = \left(\sim a_{ij}^t\right) = \left(\left(t_{ij}^t, h_{ij}^t, f_{ij}^t\right)\right)_{m \times n}(t = 1, 2, \ldots, z)$ is constructed.

The resultant target is to provide the degrees of all alternatives. After that, we will provide the stages for the solution to this issue.

Step 1: Provide the whole evaluation value of every feature for every alternative by $\widetilde{a}_{ij}^t = PFWDHM_w^{(x)}\left(\widetilde{a}_{ij}^1, \widetilde{a}_{ij}^2, \ldots, \widetilde{a}_{ij}^z\right)$ and $\widetilde{a}_{ij}^t = PFWDDHM_w^{(x)}\left(\widetilde{a}_{ij}^1, \widetilde{a}_{ij}^2, \ldots, \widetilde{a}_{ij}^z\right)$

Step 2: Evaluate the total value of each alternative taking help from the PFWDHM (PFWDDHM) operator.

$\widetilde{a}_i^t = PFWDHM_w^{(x)}(\widetilde{a}_{i1}, \widetilde{a}_{i2}, \ldots, \widetilde{a}_{in})$

and $\widetilde{a}_i^t = PFWDDHM_w^{(x)}(\widetilde{a}_{i1}, \widetilde{a}_{i2}, \ldots, \widetilde{a}_{in})$

Step 3: Evaluate the $S(\widetilde{a})$ and $H(\widetilde{a})$.

Step 4: Grade the complete alternatives $\{x_1, x_2, \ldots, x_n\}$ as select the most appropriate one.

5. Practical Example

An example is being suggested here for the explanation of this method. Take four cars $M_i = (M_1, M_2, M_3, M_4)$ and choose only one from the transportation company. We calculate all the providers from four angles $E_i = (E_1, E_2, E_3, E_4)$, and these are quality, production, service efficiency, and risk factor, and $w = (0.1, 0.4, 0.3, 0.2)^T$ represent the weight vector of attributes. Take four experts and the weight vector $(0.1, 0.4, 0.3, 0.2)^T$. $R_t = \left(a_{ij}^t\right)_{4\times 4}(t = 1, 2, 3, 4)$ show the DM given in Tables 1-4. Our ambition is to choose the best car of all.

5.1. Decision-making progress

Step 1: As E_1, E_2, E_3, and E_4 all are of the same type so we would not need to normalize the decision matrices.

Step 2: By using the PFWDHM operator on four decision matrices $R_t = (a_{ij}^t)_{m\times n}$ and develop a collective decision matrix $R = (a_{ij}^t)_{m\times n}$ which is shown in Table 5 for $x = 2$ and $\lambda = 2$.

By using the PFWDDHM operator on four decision matrices $R_t = (a_{ij}^t)_{m\times n}$ and developing a collective decision matrix $R = (a_{ij}^t)_{m\times n}$ which is shown in Table 6 for $x = 2$ and $\lambda = 2$.

Table 1
Decision matrix R_1

	E_1	E_2	E_3	E_4
M_1	(0.3, 0.3, 0.3)	(0.2, 0.4, 0.1)	(0.2, 0.3, 0.3)	(0.5, 0.1, 0.1)
M_2	(0.3, 0.4, 0.3)	(0.4, 0.2, 0.3)	(0.5, 0.1, 0.3)	(0.2, 0.2, 0.3)
M_3	(0.4, 0.2, 0.3)	(0.6, 0.1, 0.1)	(0.3, 0.3, 0.2)	(0.6, 0.2, 0.1)
M_4	(0.5, 0.1, 0.2)	(0.4, 0.2, 0.2)	(0.1, 0.4, 0.3)	(0.5, 0.3, 0.1)

Table 2
Decision matrix R_2

	E_1	E_2	E_3	E_4
M_1	(0.2, 0.1, 0.3)	(0.4, 0.4, 0.1)	(0.3, 0.2, 0.3)	(0.7, 0.1, 0.1)
M_2	(0.3, 0.4, 0.2)	(0.3, 0.1, 0.3)	(0.4, 0.1, 0.4)	(0.2, 0.4, 0.3)
M_3	(0.1, 0.3, 0.3)	(0.5, 0.1, 0.1)	(0.6, 0.1, 0.2)	(0.5, 0.3, 0.1)
M_4	(0.3, 0.1, 0.2)	(0.2, 0.2, 0.2)	(0.1, 0.4, 0.3)	(0.5, 0.2, 0.1)

Table 3
Decision matrix R_3

	E_1	E_2	E_3	E_4
M_1	(0.4, 0.3, 0.2)	(0.3, 0.4, 0.1)	(0.2, 0.4, 0.3)	(0.7, 0.1, 0.1)
M_2	(0.3, 0.4, 0.1)	(0.6, 0.1, 0.2)	(0.6, 0.1, 0.1)	(0.4, 0.1, 0.3)
M_3	(0.2, 0.2, 0.3)	(0.6, 0.1, 0.1)	(0.2, 0.2, 0.2)	(0.5, 0.1, 0.2)
M_4	(0.6, 0.1, 0.2)	(0.4, 0.3, 0.1)	(0.2, 0.4, 0.3)	(0.4, 0.3, 0.1)

Table 4
Decision matrix R_4

	E_1	E_2	E_3	E_4
M_1	(0.1, 0.3, 0.3)	(0.3, 0.3, 0.1)	(0.1, 0.4, 0.2)	(0.5, 0.1, 0.1)
M_2	(0.2, 0.4, 0.2)	(0.2, 0.2, 0.2)	(0.1, 0.2, 0.3)	(0.4, 0.2, 0.3)
M_3	(0.5, 0.1, 0.3)	(0.3, 0.1, 0.1)	(0.3, 0.1, 0.2)	(0.3, 0.2, 0.1)
M_4	(0.4, 0.1, 0.2)	(0.4, 0.3, 0.2)	(0.5, 0.1, 0.2)	(0.4, 0.3, 0.2)

Table 5
Collective decision matrix R

	G_1	G_2	G_3	G_4
A_1	(0.23, 0.27, 0.28)	(0.28, 0.38, 0.10)	(0.18, 0.33, 0.28)	(0.58, 0.10, 0.10)
A_2	(0.27, 0.40, 0.20)	(0.34, 0.36, 0.25)	(0.40, 0.12, 0.28)	(0.28, 0.20, 0.30)
A_3	(0.30, 0.19, 0.30)	(0.49, 0.10, 0.10)	(0.30, 0.17, 0.17)	(0.46, 0.19, 0.11)
A_4	(0.44, 0.10, 0.20)	(0.35, 0.25, 0.17)	(0.16, 0.35, 0.27)	(0.44, 0.28, 0.12)

Table 6
Collective decision matrix R

	G_1	G_2	G_3	G_4
A_1	$(0.24,\ 0.25,\ 0.27)$	$(0.29,\ 0.37,\ 0.10)$	$(0.19,\ 0.32,\ 0.26)$	$(0.59,\ 0.10,\ 0.10)$
A_2	$(0.27,\ 0.40,\ 0.19)$	$(0.36,\ 0.14,\ 0.24)$	$(0.44,\ 0.11,\ 0.26)$	$(0.29,\ 0.19,\ 0.30)$
A_3	$(0.30,\ 0.18,\ 0.30)$	$(0.52,\ 0.10,\ 0.10)$	$(0.31,\ 0.15,\ 0.20)$	$(0.49,\ 0.18,\ 0.11)$
A_4	$(0.46,\ 0.10,\ 0.20)$	$(0.36,\ 0.24,\ 0.17)$	$(0.17,\ 0.31,\ 0.26)$	$(0.45,\ 0.27,\ 0.11)$

Table 7
Collective decision matrix R

	G_1	G_2	G_3	G_4
A_1	$(0.53,\ 0.32,\ 0.08)$	$(0.70,\ 0.11,\ 0.18)$	$(0.30,\ 0.47,\ 0.07)$	$(0.13,\ 0.075,\ 0.05)$
A_2	$(0.72,\ 0.14,\ 0.08)$	$(0.34,\ 0.40,\ 0.18)$	$(0.17,\ 0.56,\ 0.12)$	$(0.68,\ 0.15,\ 0.09)$
A_3	$(0.84,\ 0.07,\ 0.02)$	$(0.46,\ 0.28,\ 0.08)$	$(0.04,\ 0.86,\ 0.07)$	$(0.12,\ 0.68,\ 0.08)$
A_4	$(0.26,\ 0.58,\ 0.07)$	$(0.65,\ 0.22,\ 0.11)$	$(0.81,\ 0.06,\ 0.05)$	$(0.13,\ 0.72,\ 0.07)$

Step 3: By using PFWDHM (PFWDDHM) operator to aggregate all the attributes values a_{ij}, $a'_{ij}(j = 1, 2, 3, 4)$ and get the comprehensive evaluation value for $x = 2$ and $\lambda = 2$.

$$a_1 = (0.29, 0.22, 0.24),\ a_2 = (0.34, 0.28, 0.14),$$
$$a_3 = (0.22, 0.25, 0.25),\ a_4 = (0.44, 0.19, 0.13)$$
$$a'_1 = (0.31, 0.20, 0.23),\ a'_2 = (0.37, 0.19, 0.13),$$
$$a'_3 = (0.25, 0.22, 0.23),\ a'_4 = (0.48, 0.16, 0.12).$$

Step 4: Calculate the score values.

$$S(a_1) = -0.17,\ S(a_2) = -0.08,\ S(a_3) = -0.28,\ S(a_4) = 0.12.$$
$$S(a'_1) = -0.12\ S(a'_2) = 0.05,\ S(a'_3) = -0.20,\ S(a'_4) = 0.20$$

Step 5: Rank all alternatives. $a_4 > a_2 > a_1 > a_3$. Hence, a_4 is the best choice.

By considering the different values of x, and by applying the PFWDHM (PFWDDHM) operator, we came to the result that the answer is the same.

6. Comparative Study

Keeping in view the above discussion, we will evaluate our suggested method with the previous ones. For that purpose, the following remarks show the generalization of the new operators of PFSs over IFSs and FSs.

Remarks:
1. If we put $Hx_{i_j} = 0$ in Definition 7, the PFDHM operator reduces to the IFDHM operator discussed in Li et al., (2018).
2. If we put $wHx_{i_j} = 0$ in Definition 8, the PFWDHM operator reduces to the IFWDHM operator discussed in Li et al., (2018).
3. If we put $DHx_{i_j} = 0$ in Definition 9, the PFDDHM operator reduces to the IFDDHM operator discussed in Li et al., (2018).
4. If we put $wDHx_{i_j} = 0$ in Definition 10, the PFWDDHM operator reduces to the IFWDDHM operator discussed in Li et al., (2018).
5. If we put $Hx_{i_j} = 0$ and $Fx_{i_j} = 0$ in Definition 7, the PFDHM operator reduces to the fuzzy DHM operator discussed in Dombi (1982).
6. If we put $Hx_{i_j} = 0$ and $Fx_{i_j} = 0$ in Definition 8, the PFWDHM operator reduces to the fuzzy WDHM operator discussed in Dombi (1982).
7. If we put $Hx_{i_j} = 0$ and $Fx_{i_j} = 0$ in Definition 9, the PFDDHM operator reduces to the fuzzy DDHM operator discussed in Dombi (1982).
8. If we put $Hx_{i_j} = 0$ and $Fx_{i_j} = 0$ in Definition 10, the PFWDDHM operator reduces to the fuzzy WDDHM operator discussed in Dombi (1982).

Furthermore, we require an example pertaining to MAGDM with PFNs. Keeping in view the example related to MAGDM (Liu ans Zhang, 2018), we use our suggested methods to obtain results with PFNs. Moreover, we will evaluate the results to reach the logic. Four PF decision matrices are being represented as an example (Liu ans Zhang, 2018) and to obtain the results we will use PFWFHM operator. In the application of the PFWDHM operator in Tables 1-4 given in (Liu ans Zhang, 2018), we get all feature values $a_{ij}(j = 1, 2, 3, 4)$ and get the comprehensive evaluation value for $x = 2$ and $\lambda = 2$ with the weight vector $w = (0.1, 0.4, 0.3, 0.2)^T$.

$$a_1 = (0.58, 0.24, 0.06),\ a_2 = (0.56, 0.14, 0.13),$$
$$a_3 = (0.24, 0.51, 0.07),\ a_4 = (0.68, 0.14, 0.07).$$

$S(a_1) = 28$, $S(a_2) = 0.29$, $S(a_3) = -0.34$, $S(a_4) = 0.47$. By ranking we get, $a_4 > a_2 > a_1 > a_3$. As the results are the same, therefore, it is obvious that the suggested work is applicable to PFNs too.

By comparing the results of our proposed methods with the methods proposed by Peide Liu in (Liu ans Zhang, 2018), it is clear that the results produced by Peide Liu and the results produced by us in this article are the same by solving the same example given in (Liu ans Zhang, 2018). Hence, our proposed methods are applicable and it is the generalized structure of methods proposed in Li et al., (2018).

7. Conclusion

PFS requires operators to measure the coordination of two PFSs. The most part of this article is related to the PF operators as it is the generalized production of the FSs and IFSs. Here, the HM operator, DHM operator, and DDHM operator are extended in the context of PFS to develop the PFDHM operator, PFDHM operator, PFDDHM operator, and PFWDDHM operator. Further, the properties related to the proposed operators are discussed. By using the proposed operators, the MAGDM methods are developed. Further, we applied the operators to a numerical example of a car supplier to conclude our results. A comparative study is also made to study the importance of the proposed work. In the coming years, the use of the PFNs requires to be discovered in the process of DM, risk analysis, and many other fuzzy conditions. So, in the incoming years, our aim is to develop the proposed operator in the context of interval-valued PFSs and bipolar picture FSs. It is also our aim to extend the operators in complex theory.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

Informed Consent

Informed consent was obtained from all individual participants included in the study.

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How to Cite: Saad, M., Rafiq, A., & Perez-Dominguez, L. (2022). Methods for Multiple Attribute Group Decision Making Based on Picture Fuzzy Dombi Hamy Mean Operator. *Journal of Computational and Cognitive Engineering* 00(00), 1–13, https://doi.org/10.47852/bonviewJCCE2202206