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RESEARCH ARTICLE

Analysis of Maclaurin Symmetric Mean Operators for Managing Complex Interval-Valued q-Rung Orthopair Fuzzy Setting and Their Applications

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Abstract: Risk is demonstrated as unknowns that have measurable possibilities, while complication requires unknown with no significant possibilities of the outcome. These notions are associated but are not identical. Ambiguity and risk are closely concerned notions in decision-making strategies using fuzzy set theory. Similarly, Maclaurin symmetric mean (MSM) is also massive beneficial and valuable for using to accumulate the family of attributes into a singleton set. To enhance the superiority of the research work, in this scenario, we used the informative idea of a complex interval-valued q-rung orthopair fuzzy (CIVq-ROF) setting and took a valuable tool of MSM to present the CIVq-ROF MSM (CIVq-ROFMSM), CIVq-ROF-weighted MSM (CIVq-ROFWMSM), CIVq-ROF dual MSM (CIVq-ROFDMSM), and CIVq-ROF-weighted dual MSM (CIVq-ROFWDMSM) operators. To verify the supremacy of the invented works for the different values of parameters, several specific cases are also explored. Finally, with the help of multi-attribute decision-making (MADM) skills, we identified a beneficial optimal in the presence of the source of descriptions in the form of invented operators using the decision-making process. Comparison of the invented approaches with many existing scenarios is also simplified at the end of this analysis, which shows the dominancy and competency of the diagnosed approaches.

Keywords: complex interval-valued q-rung orthopair fuzzy settings, dual Maclaurin symmetric mean operators, decision support systems

1. Introduction

Decision-making is the procedure of choosing the outstanding possible optimal in an assumed scenario. Proficient decision-making often needs you to recognize several prospective optima, imagined feasible outcomes, and several others. Handling all these and other variables can help you identify the massive proficient technique to make proficient options promptly. For this, Zadeh (1965) produced a tool called fuzzy set (FS) which becomes a valuable tool to reduce the ambiguity involved in real-life troubles such as soft set theory (Chen et al., 2010; Naveed et al., 2020) and decision-making (Chen & Tan, 1994). After it, various generalizations of FSs such as intuitionistic FS (IFS) (Atanassov, 1986), Pythagorean FS (PFS) (Yager, 2013), and q-rung orthopair FS (q-ROFS) (Yager, 2016) invented in the occurrence of truth grade (TG) $\eta'_{\mathfrak{z}}(u)$ and falsity grade (FG) $\zeta'_{\mathfrak{z}}(u)$ should have $0 \leq \eta'_{\mathfrak{z}}(u) + \zeta'_{\mathfrak{z}}(u) \leq 1, 0 \leq \eta'^{2}_{\mathfrak{z}}(u) + \zeta'^{2}_{\mathfrak{z}}(u) \leq 1$ and $0 \leq \eta'^{\mathfrak{D}}_{\mathfrak{z}}(u) + \zeta'^{\mathfrak{D}}_{\mathfrak{z}}(u) \leq 1$, where $\mathfrak{D} \geq 1$. Since its presence, intellectuals have founded distinct sorts of algorithms for evaluating the decision-making obstacles (Al-Qurashi et al., 2022; Naeem et al., 2019; Riaz et al., 2020; Tao et al., 2021; Wang et al., 2022). Later, the decision-maker knows that the role of fuzzy data become massive and broaden your horizons. As the real troubles are awkward in real life, decision-makers always suggested a simplest form to show their data in the shape of interval-valued (I-V) numbers rather than the crisp number. For this, the massive convenient and accessible tool called I-V IFS (I-VIFS) was deliberated by Atanassov (1999), which is the dominant and generalized form of I-V FS (I-VFS) (Zadeh, 1975). After completing the mathematical form of the idea of I-VIFS, several utilizations have been deliberated in different fields. Further, the idea of I-V PFS (I-VPFS) is deliberated by Garg (2016), by enhancing the quality of the mathematical form of the I-VIFS. After completing the mathematical form of the idea of I-VPFS, several utilizations have been deliberated in different fields (Akram et al., 2019; Haktanir & Kahraman, 2019; Liang et al., 2018; Peng & Li, 2019; Rahman et al., 2017). But experts have some possibilities to see what they can do. When an expert faced the data: $([0.8, 0.9], [0.4, 0.7])$, the existing I-VIFS and I-VPFS have been fruitless. The theory of I-V q-ROFS (I-Vq-ROFS), organized by Joshi et al. (2018), invented in the occurrence of TG $[\eta'^{-}_{\mathfrak{z}}(u), \eta'^{+}_{\mathfrak{z}}(u)]$ and FG $[\zeta'^{-}_{\mathfrak{z}}(u), \zeta'^{+}_{\mathfrak{z}}(u)]$ should have $0 \leq \eta'^{+\mathfrak{D}}_{\mathfrak{z}}(u) + \zeta'^{+\mathfrak{D}}_{\mathfrak{z}}(u) \leq 1$. I-Vq-ROFS is a valuable tool to reduce the ambiguity involved in real-life troubles like Archimedean Muirhead mean operators (Li et al., 2018) and decision-making (Nguyen et al., 2019; Verma & Merigó, 2019).

In the availability of the above brief discussion and decision-making strategies, it is identified that the use of existing theories is bounded to access two-dimensional data. The prevailing

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theories are continuously neglected to manage with its awkward and fluctuating information. However, we will more explain with the help of some practical examples, for instance, data got from the "medical works a software for biometric and facial recognition" changed continuously with the part of time. Because of these troubles, the fundamental theory of complex FS (CFS) was exposed by Ramot et al. (2002), Since, CFS has unable to educate anything concerning the non-supporting of the term and finally the tool constructed on them is restricted in nature. To enhance it, the key theme of complex I-VIFS (CI-VIFS) was diagnosed by Garg and Rani (2019) by generalizing the theory of complex IFS (CIFS) (Alkouri, 2012). In CI-VIFS, the TG $\eta'_{\mathfrak{z}} = \left[\eta^-_{\mathfrak{z}}, \eta^+_{\mathfrak{z}}\right] e^{\mathfrak{tl}2\pi\left(\left[\mu^-_{\eta_{\mathfrak{z}}}, \mu^+_{\eta_{\mathfrak{z}}}\right]\right)}$ and FG $\zeta'_{\mathfrak{z}} = \left[\zeta^-_{\mathfrak{z}}, \zeta^+_{\mathfrak{z}}\right] e^{\mathfrak{tl}2\pi([\mu^-_{\zeta_{\mathfrak{z}}}, \mu^+_{\zeta_{\mathfrak{z}}}])}$ satisfied the rules: $0 \leq \eta^+_{\mathfrak{z}} + \zeta^+_{\mathfrak{z}} \leq 1$ and $0 \leq \mu^+_{\eta_{\mathfrak{z}}} + \mu^+_{\zeta_{\mathfrak{z}}} \leq 1$. But for suggesting the value of TG $\eta'_{\mathfrak{z}} = [0.5, 0.6] e^{\mathfrak{tl}2\pi([0.6,0.7])}$ and FG $\zeta'_{\mathfrak{z}} = [0.4, 0.5] e^{\mathfrak{tl}2\pi([0.3,0.4])}$, the tool of CI-VIFS has been incomplete, for example, $0.6 + 0.5 = 1.1 > 1$. To enhance it, Ali et al. (2021) diagnosed the fundamental concept of complex I-VPFS (CI-VPFS) by modifying the rule of CI-VIFS, e.g., $0 \leq \eta^{+2}_{\mathfrak{z}} + \zeta^{+2}_{\mathfrak{z}} \leq 1$ and $0 \leq \mu^{+\,2}_{\eta_{\mathfrak{z}}} + \mu^{+\,2}_{\zeta_{\mathfrak{z}}} \leq 1$. The CI-VPFS, generalization of complex PFS (CPFS) (Akram & Naz, 2019), easily identifies the solution of the above problem, e.g., $0.6^2 + 0.5^2 = 0.36 + 0.25 = 0.61 < 1$. Certain implementations of CIFSs, CI-VIFSs, CPFSs, and CI-VPFSs have been diagnosed. But experts have some possibilities to see what they can do. When an expert faced the data: $\left([0.8, 0.9] e^{\mathfrak{tl}2\pi([0.8,0.9])}, [0.4, 0.7] e^{\mathfrak{tl}2\pi([0.4,0.7])}\right)$, the existing CI-VIFS and CI-VPFS have been fruitless. The theory of complex I-Vq-ROFS (CI-Vq-ROFS), organized by Garg et al. (2021), invented in the occurrence of TG $\left[\eta'^{-}_{\mathfrak{z}}(u), \eta'^{+}_{\mathfrak{z}}(u)\right] e^{\mathfrak{tl}2\pi\left(\left[\mu^-_{\eta_{\mathfrak{z}}}, \mu^+_{\eta_{\mathfrak{z}}}\right]\right)}$ and FG $\left[\zeta'^{-}_{\mathfrak{z}}(u), \zeta'^{+}_{\mathfrak{z}}(u)\right] e^{\mathfrak{tl}2\pi([\mu^-_{\zeta_{\mathfrak{z}}}, \mu^+_{\zeta_{\mathfrak{z}}}])}$ should have $0 \leq \eta'^{+\mathfrak{D}}_{\mathfrak{z}}(u) + \zeta'^{+\mathfrak{D}}_{\mathfrak{z}}(u) \leq 1$ and $0 \lesssim W^{+\mathfrak{D}}_{T_\tau}(u) + W^{+\mathfrak{D}}_{F_\tau}(u) \lesssim 1$. CI-Vq-ROFS is a valuable tool to reduce the ambiguity involved in real-life troubles such as Maclaurin symmetric mean (MSM) (Rong et al., 2020), relation using Cq-ROFSs (Nasir et al., 2021), N-soft set (Akram et al., 2021), and decision-making (Arya & Kumar, 2021). In the availability of the above discussions, we obtained that the decision-making tools contain some major issues:

1. How to utilize the interval-valued data in the field of FS theory.
2. How to obtain a lot of results from one operator.
3. How to get the best option from the collection of different decisions.

The major attention of this scenario is to provide a new tool for resolving an awkward decision-making procedure in the availability of invented approaches (MSM operators). A huge number of scholars have demonstrated very well-known ideas in the availability of FS (and their extensions) theory. But the main question is why we choose CI-Vq-ROFSs? The main reason is that the CI-Vq-ROS, generalization of I-Vq-ROFSs, and the I-Vq-ROFSs cannot contain the phase terms. Because of this reason, we can lose information during decision-making processes. To moreover demonstrate the notion of phase terms, we discuss some practical examples. Assume an intellectual wants to purchase a car based on important factors like the model and production dates of the car. Since the model and production dates of every car have changed with time, hence, choosing the best car is based on the important factors which change continuously time by time. Also, we know that such sort of data cannot handle prevailing drawbacks like FS and their extensions. But the novel concept of CI-Vq-ROFS is more suitable and accurate to manage the capable problems. The mathematical structure of CI-Vq-ROFS includes two grades in the form of complex numbers which deal with two possibilities (real and imaginary parts) at a time. The first term expressed the model and the second term showed the production data of the cars. Keeping the benefits of the above-discussed approaches, some implementations related to this work are discussed here:

1. To present the CIVq-ROF MSM (CIVq-ROFMSM), CIVq-ROF-weighted MSM (CIVq-ROFWMSM), CIVq-ROF dual MSM (CIVq-ROFDMSM), and CIVq-ROF-weighted dual MSM (CIVq-ROFWDMSM) operators.
2. To verify the supremacy of the invented works for the different values of parameters, several specific cases are also explored.
3. To identify a beneficial optimal in the presence of the source of descriptions in the form of invented operators using the decision-making process.
4. To find the comparison of the invented approaches with many existing scenarios that is also simplified at the end of this analysis, which shows the dominancy and competency of the diagnosed approaches.

The theme of this manuscript is demonstrated as follows. In Section 2, we continuously express CI-Vq-ROFSs, algebraic laws, MSM, and dual MSM (DMSM) operators. In Section 3, we use the informative idea of the CIVq-ROF setting and take a valuable tool of MSM to present the CIVq-ROFMSM, CIVq-ROFWMSM, CIVq-ROFDMSM, and CIVq-ROFWDMSM operators. To verify the supremacy of the invented works for the different values of parameters, several specific cases are also explored. In Section 4, with the help of multi-attribute decision-making (MADM) skills, we identified a beneficial optimal in the presence of the source of descriptions in the form of invented operators using the decision-making process. Comparison of the invented approaches with many existing scenarios is also simplified at the end of this analysis, which shows the dominancy and competency of the diagnosed approaches, as discussed in Section 5. Finally, we utilized the conclusion of this manuscript in Section 6.

2. Preliminaries

Some theory plays a vital role in the environment of operators, measures, and methods. Inspired by the above, we continuously express CI-Vq-ROFSs, algebraic laws, MSM, and DMSM operators in this section to develop some new operators which will be used for evaluating real-life troubles.

Definition 1: (Arya & Kumar, 2021) A CI-Vq-ROFS $\mathfrak{Rf}$ in the availability of universal set $\mho$ is simplified by:

$$\mathfrak{Rf} = \left\{ \left(u, \eta'_{\mathfrak{Rf}}(u), \zeta'_{\mathfrak{Rf}}(u) \right) : u \in \mho \right\} \quad (1)$$

By suggesting the values: $\eta'_{\mathfrak{Rf}} = \left[\eta^-_{\mathfrak{tl}}, \eta^+_{\mathfrak{tl}}\right] e^{\mathfrak{tl}2\pi\left[\mu^-_{\eta_{\mathfrak{tl}}}, \mu^+_{\eta_{\mathfrak{tl}}}\right]}$ and $\zeta'_{\mathfrak{Rf}} = [\zeta^-_{\mathfrak{tl}}, \zeta^+_{\mathfrak{tl}}] e^{\mathfrak{tl}2\pi[\mu^-_{\zeta_{\mathfrak{tl}}}, \mu^+_{\zeta_{\mathfrak{tl}}}]}$, in the availability of several rules: $0 \lesssim \eta'^{+\mathfrak{D}}_{\mathfrak{Rf}}(u) + \zeta'^{+\mathfrak{D}}_{\mathfrak{Rf}}(u) \lesssim 1$ and $0 \lesssim \mu^{+\mathfrak{D}}_{T_\tau}(u) + \mu^{+\mathfrak{D}}_{F_\tau}(u) \lesssim 1$, where $H_\tau(u) = R.e^{\mathfrak{tl}.2\pi\mu_{Rc}(u)}$ stated the refusal grade in the presence: $R = \left[\left(1 - \left(\eta^{-\mathfrak{D}}_{\mathfrak{tl}} + \zeta^{-\mathfrak{D}}_{\mathfrak{tl}}\right)\right)^{1/\mathfrak{D}}, \left(1 - \left(\eta^{+\mathfrak{D}}_{\mathfrak{tl}} + \zeta^{+\mathfrak{D}}_{\mathfrak{tl}}\right)\right)^{1/\mathfrak{D}}\right]$ and $\mu_R(u) = [(1 - (\mu^{-\mathfrak{D}}_{\eta_{\mathfrak{tl}}} + \mu^{-\mathfrak{D}}_{\zeta_{\mathfrak{tl}}}))^{1/\mathfrak{D}}, (1 - (\mu^{+\mathfrak{D}}_{\eta_{\mathfrak{tl}}} + \mu^{+\mathfrak{D}}_{\zeta_{\mathfrak{tl}}}))^{1/\mathfrak{D}}]$. The full name of CI-Vq-ROFN is simplified by: $\mathfrak{Rf} = \left([\eta^-_{\mathfrak{tl}} \eta^+_{\mathfrak{tl}}] e^{\mathfrak{tl}2\pi[\mu^-_{\eta_{\mathfrak{tl}}}, \mu^+_{\eta_{\mathfrak{tl}}}]}, [\zeta^-_{\mathfrak{tl}}, \zeta^+_{\mathfrak{tl}}] e^{\mathfrak{tl}2\pi[\mu^-_{\zeta_{\mathfrak{tl}}}, \mu^+_{\zeta_{\mathfrak{tl}}}]}\right)$.

Definition 2: (Arya & Kumar, 2021) In the availability of $\mathfrak{Rf} = \left([\eta^-_{\mathfrak{Rf}}, \eta^+_{\mathfrak{Rf}}] e^{\mathfrak{tl}2\pi[\mu^-_{\eta_{\mathfrak{Rf}}}, \mu^+_{\eta_{\mathfrak{Rf}}}]}, [\zeta^-_{\mathfrak{Rf}}, \zeta^+_{\mathfrak{Rf}}] e^{\mathfrak{tl}2\pi[\mu^-_{\zeta_{\mathfrak{Rf}}}, \mu^+_{\zeta_{\mathfrak{Rf}}}]}\right)$

and $\mathfrak{I} = \left([\eta_{\mathfrak{I}}^{-}, \eta_{\mathfrak{I}}^{+}]e^{\mathfrak{tl}2\pi[\mu_{\eta_{\mathfrak{I}}}^{-}, \mu_{\eta_{\mathfrak{I}}}^{+}]}, [\zeta_{\mathfrak{I}}^{-}, \zeta_{\mathfrak{I}}^{+}]e^{\mathfrak{tl}2\pi[\mu_{\zeta_{\mathfrak{I}}}^{-}, \mu_{\zeta_{\mathfrak{I}}}^{+}]}\right)$, shown CI-Vq-ROFNs, we get

$$\mathfrak{Rf} \subseteq \mathfrak{I} \text{ if } \eta_{\mathfrak{Rf}}^{-} \lesssim \eta_{\mathfrak{I}}^{-}, \zeta_{\mathfrak{Rf}}^{-} \geq \zeta_{\mathfrak{I}}^{-}, \eta_{\mathfrak{Rf}}^{+} \lesssim \eta_{\mathfrak{I}}^{+}, \zeta_{\mathfrak{Rf}}^{+} \geq \zeta_{\mathfrak{I}}^{+} \text{ and } \mu_{\eta_{\mathfrak{Rf}}}^{-} \lesssim \mu_{\eta_{\mathfrak{I}}}^{-}, \mu_{\zeta_{\mathfrak{Rf}}}^{-} \geq \mu_{\zeta_{\mathfrak{I}}}^{-}, \mu_{\eta_{\mathfrak{Rf}}}^{+} \lesssim \mu_{\eta_{\mathfrak{I}}}^{+}, \mu_{\zeta_{\mathfrak{Rf}}}^{+} \geq \mu_{\zeta_{\mathfrak{I}}}^{+} \tag{2}$$

$$\mathfrak{Rf} = \mathfrak{I} \text{ iff } \eta_{\mathfrak{Rf}}^{-} = \eta_{\mathfrak{I}}^{-}, \zeta_{\mathfrak{Rf}}^{-} = \zeta_{\mathfrak{I}}^{-}, \eta_{\mathfrak{Rf}}^{+} = \eta_{\mathfrak{I}}^{+}, \zeta_{\mathfrak{Rf}}^{+} = \zeta_{\mathfrak{I}}^{+} \text{ and } \mu_{\eta_{\mathfrak{Rf}}}^{-} = \mu_{\eta_{\mathfrak{I}}}^{-}, \mu_{\zeta_{\mathfrak{Rf}}}^{-} = \mu_{\zeta_{\mathfrak{I}}}^{-}, \mu_{\eta_{\mathfrak{Rf}}}^{+} = \mu_{\eta_{\mathfrak{I}}}^{+}, \mu_{\zeta_{\mathfrak{Rf}}}^{+} = \mu_{\zeta_{\mathfrak{I}}}^{+} \tag{3}$$

$$\mathfrak{Rf}^{c} = \left|\left(\left[\zeta_{\mathfrak{Rf}}^{-}, \zeta_{\mathfrak{Rf}}^{+}\right]e^{\mathfrak{tl}2\pi\left[\mu_{\zeta_{\mathfrak{Rf}}}^{-}, \mu_{\zeta_{\mathfrak{Rf}}}^{+}\right]}, \left[\eta_{\mathfrak{Rf}}^{-}, \eta_{\mathfrak{Rf}}^{+}\right]e^{\mathfrak{tl}2\pi\left[\mu_{\eta_{\mathfrak{Rf}}}^{-}, \mu_{\eta_{\mathfrak{Rf}}}^{+}\right]}\right)\right. \tag{4}$$

Definition 3: (Ali et al., 2020) In the availability of $\mathfrak{Rf} = \left([\eta_{\mathfrak{Rf}}^{-}, \eta_{\mathfrak{Rf}}^{+}]e^{\mathfrak{tl}2\pi[\mu_{\eta_{\mathfrak{Rf}}}^{-}, \mu_{\eta_{\mathfrak{Rf}}}^{+}]}, [\zeta_{\mathfrak{Rf}}^{-}, \zeta_{\mathfrak{Rf}}^{+}]e^{\mathfrak{tl}2\pi[\mu_{\zeta_{\mathfrak{Rf}}}^{-}, \mu_{\zeta_{\mathfrak{Rf}}}^{+}]}\right)$, the score function (SF) S and accuracy function (AF) H are simplified by:

$$S(\mathfrak{Rf}) = \frac{1}{4}\left|\eta_{\mathfrak{Rf}}^{-\mathfrak{D}} - \zeta_{\mathfrak{Rf}}^{-\mathfrak{D}} + \mu_{\eta_{\mathfrak{Rf}}}^{-\mathfrak{D}} - \mu_{\zeta_{\mathfrak{Rf}}}^{-\mathfrak{D}} + \eta_{\mathfrak{Rf}}^{+\mathfrak{D}} - \zeta_{\mathfrak{Rf}}^{+\mathfrak{D}} + \mu_{\eta_{\mathfrak{Rf}}}^{+\mathfrak{D}} - \mu_{\zeta_{\mathfrak{Rf}}}^{+\mathfrak{D}}\right| \tag{5}$$

$$H(\mathfrak{Rf}) = \frac{1}{4}\left|\eta_{\mathfrak{Rf}}^{-\mathfrak{D}} + \zeta_{\mathfrak{Rf}}^{-\mathfrak{D}} + \mu_{\eta_{\mathfrak{Rf}}}^{-\mathfrak{D}} + \mu_{\zeta_{\mathfrak{Rf}}}^{-\mathfrak{D}} + \eta_{\mathfrak{Rf}}^{+\mathfrak{D}} + \zeta_{\mathfrak{Rf}}^{+\mathfrak{D}} + \mu_{\eta_{\mathfrak{Rf}}}^{+\mathfrak{D}} + \mu_{\zeta_{\mathfrak{Rf}}}^{+\mathfrak{D}}\right| \tag{6}$$

It is noticed that $S(\mathfrak{Rf}), H(\mathfrak{Rf}) \in [-1, 1]$. In the availability of Eqs. (5) and (6), we get

1. $\mathfrak{Rf} > \mathfrak{I}$ if $S(\mathfrak{Rf}) > S(\mathfrak{I})$ or $H(\mathfrak{Rf}) > H(\mathfrak{I})$.
2. $\mathfrak{Rf} < \mathfrak{I}$ if $S(\mathfrak{Rf}) < S(\mathfrak{I})$ or $H(\mathfrak{Rf}) < H(\mathfrak{I})$.
3. $\mathfrak{Rf} = \mathfrak{I}$ if $S(\mathfrak{Rf}) = S(\mathfrak{I})$ or $H(\mathfrak{Rf}) = H(\mathfrak{I})$.

Definition 4: (Ali et al., 2020) In the availability of $\mathfrak{Rf} = \left([\eta_{\mathfrak{Rf}}^{-}, \eta_{\mathfrak{Rf}}^{+}]e^{\mathfrak{tl}2\pi[\mu_{\eta_{\mathfrak{Rf}}}^{-}, \mu_{\eta_{\mathfrak{Rf}}}^{+}]}, [\zeta_{\mathfrak{Rf}}^{-}, \zeta_{\mathfrak{Rf}}^{+}]e^{\mathfrak{tl}2\pi[\mu_{\zeta_{\mathfrak{Rf}}}^{-}, \mu_{\zeta_{\mathfrak{Rf}}}^{+}]}\right)$ and $\mathfrak{I} = \left([\eta_{\mathfrak{I}}^{-}, \eta_{\mathfrak{I}}^{+}]e^{\mathfrak{tl}2\pi[\mu_{\eta_{\mathfrak{I}}}^{-}, \mu_{\eta_{\mathfrak{I}}}^{+}]}, [\zeta_{\mathfrak{I}}^{-}, \zeta_{\mathfrak{I}}^{+}]e^{\mathfrak{tl}2\pi[\mu_{\zeta_{\mathfrak{I}}}^{-}, \mu_{\zeta_{\mathfrak{I}}}^{+}]}\right)$, shown CI-Vq-ROFNs with $\gamma > 0$, we get

$$\mathfrak{Rf} \vee \mathfrak{I} = \left(\begin{array}{c}\left[max\left(\eta_{\mathfrak{Rf}}^{-}, \eta_{\mathfrak{I}}^{-}\right), max\left(\eta_{\mathfrak{Rf}}^{+}, \eta_{\mathfrak{I}}^{+}\right)\right]e^{\mathfrak{tl}.2\pi\left[max\left(\mu_{\eta_{\mathfrak{Rf}}}^{-}, \mu_{\eta_{\mathfrak{I}}}^{-}\right), max\left(\mu_{\eta_{\mathfrak{Rf}}}^{+}, \mu_{\eta_{\mathfrak{I}}}^{+}\right)\right]}, \\ \left[mi\mathfrak{p}\left(\zeta_{\mathfrak{Rf}}^{-}, \zeta_{\mathfrak{I}}^{-}\right), mi\mathfrak{p}\left(\zeta_{\mathfrak{Rf}}^{+}, \zeta_{\mathfrak{I}}^{+}\right)\right]e^{\mathfrak{tl}.2\pi\left[m\mathfrak{tl}\mathfrak{p}\left(\mu_{\zeta_{\mathfrak{Rf}}}^{-}, \mu_{\zeta_{\mathfrak{I}}}^{-}\right), m\mathfrak{tl}\mathfrak{p}\left(\mu_{\zeta_{\mathfrak{Rf}}}^{+}, \mu_{\zeta_{\mathfrak{I}}}^{+}\right)\right]}\end{array}\right) \tag{7}$$

$$\mathfrak{Rf} \wedge \mathfrak{I} = \left(\begin{array}{c}\left[min\left(\eta_{\mathfrak{Rf}}^{-}, \eta_{\mathfrak{I}}^{-}\right), min\left(\eta_{\mathfrak{Rf}}^{+}, \eta_{\mathfrak{I}}^{+}\right)\right]e^{\mathfrak{tl}.2\pi\left[min\left(\mu_{\eta_{\mathfrak{Rf}}}^{-}, \mu_{\eta_{\mathfrak{I}}}^{-}\right), min\left(\mu_{\eta_{\mathfrak{Rf}}}^{+}, \mu_{\eta_{\mathfrak{I}}}^{+}\right)\right]}, \\ \left[max\left(\zeta_{\mathfrak{Rf}}^{-}, \zeta_{\mathfrak{I}}^{-}\right), max\left(\zeta_{\mathfrak{Rf}}^{+}, \zeta_{\mathfrak{I}}^{+}\right)\right]e^{\mathfrak{tl}.2\pi\left[max\left(\mu_{\zeta_{\mathfrak{Rf}}}^{-}, \mu_{\zeta_{\mathfrak{I}}}^{-}\right), max\left(\mu_{\zeta_{\mathfrak{Rf}}}^{+}, \mu_{\zeta_{\mathfrak{I}}}^{+}\right)\right]}\end{array}\right) \tag{8}$$

$$\mathfrak{Rf} \oplus \mathfrak{I} = \left(\begin{array}{c}\left[\left(\eta_{\mathfrak{Rf}}^{-\mathfrak{D}} + \eta_{\mathfrak{I}}^{-\mathfrak{D}} - \eta_{\mathfrak{Rf}}^{-\mathfrak{D}}\eta_{\mathfrak{I}}^{-\mathfrak{D}}\right)^{\frac{1}{\mathfrak{D}}}, \left(\eta_{\mathfrak{Rf}}^{+\mathfrak{D}} + \eta_{\mathfrak{I}}^{+\mathfrak{D}} - \eta_{\mathfrak{Rf}}^{+\mathfrak{D}}\eta_{\mathfrak{I}}^{+\mathfrak{D}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\mathfrak{tl}.2\pi\left[\left(\mu_{\eta_{\mathfrak{Rf}}}^{-\mathfrak{D}} + \mu_{\eta_{\mathfrak{I}}}^{-\mathfrak{D}} - \mu_{\eta_{\mathfrak{Rf}}}^{-\mathfrak{D}}\mu_{\eta_{\mathfrak{I}}}^{-\mathfrak{D}}\right)^{\frac{1}{\mathfrak{D}}}, \left(\mu_{\eta_{\mathfrak{Rf}}}^{+\mathfrak{D}} + \mu_{\eta_{\mathfrak{I}}}^{+\mathfrak{D}} - \mu_{\eta_{\mathfrak{Rf}}}^{+\mathfrak{D}}\mu_{\eta_{\mathfrak{I}}}^{+\mathfrak{D}}\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\zeta_{\mathfrak{Rf}}^{-}\zeta_{\mathfrak{I}}^{-}, \zeta_{\mathfrak{Rf}}^{+}\zeta_{\mathfrak{I}}^{+}\right]e^{\mathfrak{tl}.2\pi\left[\mu_{\zeta_{\mathfrak{Rf}}}^{-}\mu_{\zeta_{\mathfrak{I}}}^{-}, \mu_{\zeta_{\mathfrak{Rf}}}^{+}\mu_{\zeta_{\mathfrak{I}}}^{+}\right]}\end{array}\right) \tag{9}$$

$$\mathfrak{Rf} \otimes \mathfrak{I} = \left(\begin{array}{c}\left[\eta_{\mathfrak{Rf}}^{-}\eta_{\mathfrak{I}}^{-}, \eta_{\mathfrak{Rf}}^{+}\eta_{\mathfrak{I}}^{+}\right]e^{\mathfrak{tl}.2\pi\left[\mu_{\eta_{\mathfrak{Rf}}}^{-}\mu_{\eta_{\mathfrak{I}}}^{-}, \mu_{\eta_{\mathfrak{Rf}}}^{+}\mu_{\eta_{\mathfrak{I}}}^{+}\right]}, \\ \left[\left(\zeta_{\mathfrak{Rf}}^{-\mathfrak{D}} + \zeta_{\mathfrak{I}}^{-\mathfrak{D}} - \zeta_{\mathfrak{Rf}}^{-\mathfrak{D}}\zeta_{\mathfrak{I}}^{-\mathfrak{D}}\right)^{\frac{1}{\mathfrak{D}}}, \left(\zeta_{\mathfrak{Rf}}^{+\mathfrak{D}} + \zeta_{\mathfrak{I}}^{+\mathfrak{D}} - \zeta_{\mathfrak{Rf}}^{+\mathfrak{D}}\zeta_{\mathfrak{I}}^{+\mathfrak{D}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\mathfrak{tl}.2\pi\left[\left(\mu_{\zeta_{\mathfrak{Rf}}}^{-\mathfrak{D}} + \mu_{\zeta_{\mathfrak{I}}}^{-\mathfrak{D}} - \mu_{\eta_{\mathfrak{Rf}}}^{-\mathfrak{D}}\mu_{\eta_{\mathfrak{I}}}^{-\mathfrak{D}}\right)^{\frac{1}{\mathfrak{D}}}, \left(\mu_{\zeta_{\mathfrak{Rf}}}^{+\mathfrak{D}} + \mu_{\zeta_{\mathfrak{I}}}^{+\mathfrak{D}} - \mu_{\eta_{\mathfrak{Rf}}}^{+\mathfrak{D}}\mu_{\eta_{\mathfrak{I}}}^{+\mathfrak{D}}\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right) \tag{10}$$

$$\gamma\mathfrak{Rf} = \left(\begin{array}{c}\left[\left(1 - \left(1 - \eta_{\mathfrak{Rf}}^{-\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(1 - \eta_{\mathfrak{Rf}}^{+\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}\right]e^{\mathfrak{tl}.2\pi\left[\left(1 - \left(1 - \mu_{\eta_{\mathfrak{Rf}}}^{-\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(1 - \mu_{\eta_{\mathfrak{Rf}}}^{+\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\zeta_{\mathfrak{Rf}}^{-\gamma}, \zeta_{\mathfrak{Rf}}^{+\gamma}\right]e^{\mathfrak{tl}.2\pi\left[\mu_{\zeta_{\mathfrak{Rf}}}^{-\gamma}, \mu_{\zeta_{\mathfrak{Rf}}}^{+\gamma}\right]}\end{array}\right) \tag{11}$$

$$\mathfrak{Rf}^{\gamma} = \left(\begin{array}{c}c\left[\eta_{\mathfrak{Rf}}^{-\gamma}, \eta_{\mathfrak{Rf}}^{+\gamma}\right]e^{\mathfrak{tl}.2\pi\left[\mu_{\eta_{\mathfrak{Rf}}}^{-\gamma}, \mu_{\eta_{\mathfrak{Rf}}}^{+\gamma}\right]}, \\ \left[\left(1 - \left(1 - \zeta_{\mathfrak{Rf}}^{-\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(1 - \zeta_{\mathfrak{Rf}}^{+\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}\right]e^{\mathfrak{tl}.2\pi\left[\left(1 - \left(1 - \mu_{\zeta_{\mathfrak{Rf}}}^{-\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(1 - \mu_{\zeta_{\mathfrak{Rf}}}^{+\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right) \tag{12}$$

Definition 5: (Rong et al., 2020) In the availability of any positive integer $\mathfrak{I}_{\mathfrak{tl}}, \mathfrak{tl} = 1, 2, 3, .., \mathfrak{p}$, the MSM operator is simplified by:

$$MSM^{\ell}\left(\mathfrak{I}_{1}, \mathfrak{I}_{2}, \mathfrak{I}_{3}, .., \mathfrak{I}_{\mathfrak{p}}\right) = \left(\left(\frac{\sum_{1 \lesssim \mathfrak{tl}_{1} \lesssim \mathfrak{tl}_{2} \lesssim .. \lesssim \mathfrak{tl}_{\ell} \lesssim \mathfrak{p}}\left(\prod_{\mathfrak{tl}=1}^{\ell}\mathfrak{I}_{\mathfrak{tl}\mathfrak{tl}}\right)}{C_{\mathfrak{p}}^{\ell}}\right)\right)^{\frac{1}{\ell}} \tag{13}$$

With various properties:

$$MSM^{\ell}(0, 0, .., 0) = 0$$

$$MSM^{\ell}(a, a, .., a) = a$$

$$MSM^{\ell}\left(\mathfrak{I}_{1}, \mathfrak{I}_{2}, \mathfrak{I}_{3}, .., \mathfrak{I}_{\mathfrak{p}}\right) \lesssim MSM^{\ell}\left(b_{1}, b_{2}, b_{3}, .., b_{\mathfrak{p}}\right) \text{if } a_{\mathfrak{tl}} \lesssim b_{\mathfrak{tl}}, \mathfrak{tl} = 1, 2, .., \mathfrak{p}.$$

Definition 6: (Rong et al., 2020) In the availability of any positive integer $\mathfrak{I}_{\mathfrak{tl}}, \mathfrak{tl} = 1, 2, 3, .., \mathfrak{p}$, the DMSM operator is simplified by:

$$DMSM^{\ell}\left(\mathfrak{I}_{1}, \mathfrak{I}_{2}, \mathfrak{I}_{3}, .., \mathfrak{I}_{\mathfrak{p}}\right) = \frac{1}{\ell}\left(\prod_{1 \lesssim \mathfrak{tl}_{1} \lesssim \mathfrak{tl}_{2} \lesssim .. \lesssim \mathfrak{tl}_{\ell} \lesssim \mathfrak{p}}\left(\sum_{\mathfrak{tl}=1}^{\ell}\mathfrak{I}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right) \tag{14}$$

where $C_{\mathfrak{p}}^{\ell}$ is the binomial coefficent.

3. MSM/DMSM Operators for CI-Vq-ROF Settings

Risk is demonstrated as unknowns that have measurable possibilities, while complication requires unknown with no significant possibilities of the outcome. These notions are associated but are not identical. Ambiguity and risk are closely concerned notions in decision-making strategies using FS theory. Similarly, MSM is also massive beneficial and valuable for using to accumulate the family of attributes into a one-term set. To enhance the superiority of the research works, in this scenario, we used the informative idea of the CIVq-ROF setting and took a valuable tool of MSM to present the CIVq-ROFMSM, CIVq-ROFWMSM, CIVq-ROFDMSM, and CIVq-ROFWDMSM operators. To verify the supremacy of the invented works for the different values of parameters, several specific cases are also explored. Further, the mathematical terms $\mathbb{W}_{\mathfrak{tl}} \in [0,1]$, $\mathbb{W} = (\mathbb{W}_1, \mathbb{W}_2, .., \mathbb{W}_{\mathfrak{p}})^T$ with $\sum_{\mathfrak{tl}=1}^{\mathfrak{p}} \mathbb{W}_{\mathfrak{tl}} = 1$ stated the weight vectors. The CI-Vq-ROFNs are denoted by: $\Re\mathfrak{f} = \left([\eta_{\mathfrak{tl}}^-, \eta_{\mathfrak{tl}}^+] e^{\mathfrak{tl}2\pi[\mu_{\eta_{\mathfrak{tl}}}^-, \mu_{\eta_{\mathfrak{tl}}}^+]}, [\zeta_{\mathfrak{tl}}^-, \zeta_{\mathfrak{tl}}^+] e^{\mathfrak{tl}2\pi\left[\mu_{\zeta_{\mathfrak{tl}}}^-, \mu_{\zeta_{\mathfrak{tl}}}^+\right]}\right)$.

Definition 7: The CI-Vq-ROFMSM operator is simplified by:

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, .., \mathfrak{I}_{\mathfrak{p}}) = \left(\left(\frac{\oplus_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}} \left(\otimes_{\mathfrak{tl}=1}^{\ell} \mathfrak{I}_{\mathfrak{tl}\mathfrak{tl}}\right)}{C_{\mathfrak{p}}^{\ell}}\right)\right)^{\frac{1}{\ell}} \tag{15}$$

Theorem 1: In the availability of Eq. (15), we get

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, .., \mathfrak{I}_{\mathfrak{p}}) = \left(\begin{array}{c} \left[\begin{array}{c} \left(\left(1 - \left(\prod_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}} \left(1 - \left(\prod_{\mathfrak{tl}=1}^{\ell} \eta_{\mathfrak{tl}\mathfrak{tl}}^-\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}, \\ \left(\left(1 - \left(\prod_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}} \left(1 - \left(\prod_{\mathfrak{tl}=1}^{\ell} \eta_{\mathfrak{tl}\mathfrak{tl}}^+\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}} \end{array}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(\prod_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell} \mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^-\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}, \left(\left(1-\left(\prod_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell} \mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^+\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]}, \\ \left[\begin{array}{c} \left(1 - \left(1 - \left(\prod_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}} \left(1 - \prod_{\mathfrak{tl}=1}^{\ell}\left(1 - (\zeta_{\mathfrak{tl}\mathfrak{tl}}^-)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}, \\ \left(1 - \left(1 - \left(\prod_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}} \left(1 - \prod_{\mathfrak{tl}=1}^{\ell}\left(1 - (\zeta_{\mathfrak{tl}\mathfrak{tl}}^+)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}} \end{array}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^-\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(\prod_{1 \lesssim \mathfrak{tl}_1 \lesssim \mathfrak{tl}_2 \lesssim .., \lesssim \mathfrak{tl}_\ell \lesssim \mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^+\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right]} \end{array}\right) \tag{16}$$

Proof: In the presence of Eq. (9) to Eq. (12), we get

$$\otimes_{\mathfrak{tl}=1}^{\ell} \mathfrak{I}_{\mathfrak{tl}\mathfrak{tl}} = \left(\begin{array}{c} \left[\left(\prod_{\mathfrak{tl}=1}^{\ell} \eta_{\mathfrak{tl}\mathfrak{tl}}^-\right), \left(\prod_{\mathfrak{tl}=1}^{\ell} \eta_{\mathfrak{tl}\mathfrak{tl}}^+\right)\right] e^{\mathfrak{tl}2\pi\left[\prod_{\mathfrak{tl}=1}^{\ell} \mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^-, \prod_{\mathfrak{tl}=1}^{\ell} \mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^+\right]}, \\ \left[\left(1 - \prod_{\mathfrak{tl}=1}^{\ell}\left(1 - (\zeta_{\mathfrak{tl}\mathfrak{tl}}^-)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \prod_{\mathfrak{tl}=1}^{\ell}\left(1 - (\zeta_{\mathfrak{tl}\mathfrak{tl}}^+)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^-\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^+\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]} \end{array}\right)$$

$$\oplus_{1\lesssim\mathfrak{tl}_1\lesssim_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(\otimes_{\mathfrak{tl}=1}^{\ell}\Im_{\mathfrak{tl}\mathfrak{tl}}\right)$$

$$=\left(\begin{array}{c}\left[\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\mu^{-}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\mu^{+}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]},\\ \left[\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-(\zeta^{-}_{\mathfrak{tl}\mathfrak{tl}})^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-(\zeta^{+}_{\mathfrak{tl}\mathfrak{tl}})^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right)$$

Then, we get

$$\frac{\oplus_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(\otimes_{\mathfrak{tl}=1}^{\mathfrak{p}}\Im_{\mathfrak{tl}\mathfrak{tl}}\right)}{C_{\mathfrak{p}}^{\ell}}$$

$$=\left(\begin{array}{c}\left[\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\mu^{-}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\mu^{+}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right]},\\ \left[\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-(\zeta^{-}_{\mathfrak{tl}\mathfrak{tl}})^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}},\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-(\zeta^{+}_{\mathfrak{tl}\mathfrak{tl}})^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}},\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right]}\end{array}\right)$$

$$\text{CI-Vq-ROFMSM}^{\ell}\left(\Im_1,\Im_2,\Im_3,..,\Im_{\mathfrak{p}}\right)$$

$$=\left(\begin{array}{c}\left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\mu^{-}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\mu^{+}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]},\\ \left[\begin{array}{c}\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-(\zeta^{-}_{\mathfrak{tl}\mathfrak{tl}})^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\\ \left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-(\zeta^{+}_{\mathfrak{tl}\mathfrak{tl}})^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\end{array}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right)$$

Example 1: Assume $\Im_1=\left([0.301,0.401]e^{\mathfrak{tl}2\pi[0.301,0.401]},[0.601,0.701]e^{\mathfrak{tl}2\pi[0.601,0.701]}\right)$, $\Im_2=\left([0.401,0.501]e^{\mathfrak{tl}2\pi[0.401,0.501]},[0.301,0.401]e^{\mathfrak{tl}2\pi[0.301,0.401]}\right)$, $\Im_3=\left([0.701,0.801]e^{\mathfrak{tl}2\pi[0.701,0.801]},[0.201,0.301]\ e^{\mathfrak{tl}2\pi[0.201,0.301]}\right)$, and $\Im_4=\left([0.501,0.601]e^{\mathfrak{tl}2\pi[0.501,0.601]},[0.201,\ 0.301]\ e^{\mathfrak{tl}2\pi[0.201,0.301]}\right)$ for $\ell=2$ and $\mathfrak{D}=2$, then by using Eq. (16), we have

$$\Im_1\otimes\Im_2$$

$$=\left(\begin{array}{c}[0.301*0.401,0.401*0.501]e^{\mathfrak{tl}2\pi[0.301*0.401,0.401*0.501]},\\ \left[\begin{array}{c}c(0.601^2+0.301^2-0.601^2+0.301^2)^{\frac{1}{2}},\\ (0.701^2+0.401^2-0.701^2+0.401^2)^{\frac{1}{2}}\end{array}\right]e^{\mathfrak{tl}2\pi[0.601,0.701]}\end{array}\right)$$

$$=\left([0.1201,0.2001]e^{\mathfrak{tl}2\pi[0.1201,0.2001]},[0.6463,0.7561]e^{\mathfrak{tl}2\pi[0.6463,0.7561]}\right)$$

$$\mathfrak{I}_1 \otimes \mathfrak{I}_3 = \left([0.2101, 0.3201]e^{\mathfrak{tl}2\pi[0.2101,0.3201]}, [0.6211, 0.7322]e^{\mathfrak{tl}2\pi[0.6211,0.7322]}\right)$$

$$\mathfrak{I}_1 \otimes \mathfrak{I}_4 = \left([0.1501, 0.2401]e^{\mathfrak{tl}2\pi[0.1501,0.2401]}, [0.6211, 0.7322]e^{\mathfrak{tl}2\pi[0.6211,0.7322]}\right)$$

$$\mathfrak{I}_2 \otimes \mathfrak{I}_3 = \left([0.2801, 0.4001]e^{\mathfrak{tl}2\pi[0.2801,0.4001]}, [0.3556, 0.4855]e^{\mathfrak{tl}2\pi[0.3556,0.4855]}\right)$$

$$\mathfrak{I}_2 \otimes \mathfrak{I}_4 = \left([0.2001, 0.3001]e^{\mathfrak{tl}2\pi[0.2001,0.3001]}, [0.3556, 0.4855]e^{\mathfrak{tl}2\pi[0.3556,0.4855]}\right)$$

$$\mathfrak{I}_3 \otimes \mathfrak{I}_4 = \left([0.3501, 0.4801]e^{\mathfrak{tl}2\pi[0.3501,0.4801]}, [0.2801, 0.4147]e^{\mathfrak{tl}2\pi[0.2801,0.4147]}\right)$$

Then by using Eq. (15), we have

$$CI\text{-}Vq\text{-}ROFMSM^2(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \mathfrak{I}_4) = \left(\left(\frac{\oplus_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim 4}\left(\mathfrak{I}_{\mathfrak{tl}_1} \otimes \mathfrak{I}_{\mathfrak{tl}_2}\right)}{C_4^2}\right)\right)^{\frac{1}{2}}$$

$$= \left([0.4830, 0.5835]e^{\mathfrak{tl}2\pi[0.4830,0.5835]}, [0.5114, 0.5958]e^{\mathfrak{tl}2\pi[0.5114,0.5958]}\right).$$

Theorem 2: (Idempotency) If $\mathfrak{I}_{\mathfrak{tl}} = \left(\left[\eta_{\mathfrak{tl}}^-, \eta_{\mathfrak{tl}}^+\right]e^{\mathfrak{tl}2\pi\left[\mu_{\eta_{\mathfrak{tl}}}^-, \mu_{\eta_{\mathfrak{tl}}}^+\right]}, \left[\zeta_{\mathfrak{tl}}^-, \zeta_{\mathfrak{tl}}^+\right]e^{\mathfrak{tl}2\pi\left[\mu_{\zeta_{\mathfrak{tl}}}^-, \mu_{\zeta_{\mathfrak{tl}}}^+\right]}\right) = \mathfrak{I}$, $\mathfrak{tl} = 1, 2, 3, .., \mathfrak{p}$, then

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, .., \mathfrak{I}_{\mathfrak{p}}\right) = \mathfrak{I} \quad (17)$$

Proof: If $\mathfrak{I}_{\mathfrak{tl}} = \left(\left[\eta_{\mathfrak{tl}}^-, \eta_{\mathfrak{tl}}^+\right]e^{\mathfrak{tl}2\pi\left[\mu_{\eta_{\mathfrak{tl}}}^-, \mu_{\eta_{\mathfrak{tl}}}^+\right]}, \left[\zeta_{\mathfrak{tl}}^-, \zeta_{\mathfrak{tl}}^+\right]e^{\mathfrak{tl}2\pi\left[\mu_{\zeta_{\mathfrak{tl}}}^-, \mu_{\zeta_{\mathfrak{tl}}}^+\right]}\right) = \mathfrak{I}$, then we get

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, .., \mathfrak{I}_{\mathfrak{p}}\right)$$

$$= \left(\begin{array}{c} \left[\left(\left(1 - \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1 - \left(\prod_{\mathfrak{tl}=1}^{\ell}\eta_{\mathfrak{tl}\mathfrak{tl}}^-\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}, \left(\left(1 - \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1 - \left(\prod_{\mathfrak{tl}=1}^{\ell}\eta_{\mathfrak{tl}\mathfrak{tl}}^+\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^-\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}, \left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^+\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]}, \\ \left[\begin{array}{c} \left(1 - \left(1 - \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1 - \prod_{\mathfrak{tl}=1}^{\ell}\left(1 - \left(\zeta_{\mathfrak{tl}\mathfrak{tl}}^-\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}, \\ \left(1 - \left(1 - \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1 - \prod_{\mathfrak{tl}=1}^{\ell}\left(1 - \left(\zeta_{\mathfrak{tl}\mathfrak{tl}}^+\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}} \end{array}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^-\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^+\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right]} \end{array}\right)$$

$$= \left(\begin{array}{c} \left[\left(\left(1 - \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1 - \left(\eta_{\mathfrak{I}}^-\right)^{\mathfrak{D}\ell}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}, \left(\left(1 - \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1 - \left(\eta_{\mathfrak{I}}^+\right)^{\mathfrak{D}\ell}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\mu_{\eta_{\mathfrak{I}}}^-\right)^{\mathfrak{D}\ell}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}, \left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\mu_{\eta_{\mathfrak{I}}}^+\right)^{\mathfrak{D}\ell}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]}, \\ \left[\left(1 - \left(1 - \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1 - \left(\zeta_{\mathfrak{I}}^-\right)^{\mathfrak{D}}\right)^{\ell}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(1 - \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1 - \left(\zeta_{\mathfrak{I}}^+\right)^{\mathfrak{D}}\right)^{\ell}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\mu_{\zeta_{\mathfrak{I}}}^-\right)^{\mathfrak{D}}\right)^{\ell}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\mu_{\zeta_{\mathfrak{I}}}^+\right)^{\mathfrak{D}}\right)^{\ell}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right]} \end{array}\right)$$

$$= \begin{pmatrix} \left[\left(\left(1-\left(\left(1-\left(\eta_{\mathfrak{J}}^{-}\right)^{\mathfrak{D}\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}, \left(\left(1-\left(\left(1-\left(\eta_{\mathfrak{J}}^{+}\right)^{\mathfrak{D}\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}\right] \\ e^{\imath 2\pi\left[\left(\left(1-\left(\left(1-\left(\mu_{\eta_{\mathfrak{J}}}^{-}\right)^{\mathfrak{D}\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}, \left(\left(1-\left(\left(1-\left(\mu_{\eta_{\mathfrak{J}}}^{+}\right)^{\mathfrak{D}\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}\right]}, \\ \left[\left(1-\left(1-\left(\left(1-\left(1-\left(\zeta_{\mathfrak{J}}^{-}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(\left(1-\left(1-\left(\zeta_{\mathfrak{J}}^{+}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\imath 2\pi\left[\left(1-\left(1-\left(\left(1-\left(1-\left(\mu_{\zeta_{\mathfrak{J}}}^{-}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(\left(1-\left(1-\left(\mu_{\zeta_{\mathfrak{J}}}^{+}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}\right]} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\left(\left(1-\left(\left(1-\left(_{\mathfrak{J}}^{-}\right)^{\mathfrak{D}\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}, \left(\left(1-\left(\left(1-\left(\eta_{\mathfrak{J}}^{+}\right)^{\mathfrak{D}\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}\right] \\ e^{\imath 2\pi\left[\left(\left(1-\left(\left(1-\left(\mu_{\eta_{\mathfrak{J}}}^{-}\right)^{\mathfrak{D}\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}, \left(\left(1-\left(\left(1-\left(\mu_{\eta_{\mathfrak{J}}}^{+}\right)^{\mathfrak{D}\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}\right]}, \\ \left[\left(1-\left(1-\left(\left(1-\left(1-\left(\zeta_{\mathfrak{J}}^{-}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(\left(1-\left(1-\left(\zeta_{\mathfrak{J}}^{+}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\imath 2\pi\left[\left(1-\left(1-\left(\left(1-\left(1-\left(\mu_{\zeta_{\mathfrak{J}}}^{-}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(\left(1-\left(1-\left(\mu_{\zeta_{\mathfrak{J}}}^{+}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{C_{\mathrm{p}}^{\mathscr{F}}}}\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}\right]} \end{pmatrix}$$

$$= \begin{pmatrix} \left[\left(\left(1-\left(1-\left(\eta_{\mathfrak{J}}^{-}\right)^{\mathfrak{D}\mathscr{F}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}, \left(\left(1-\left(1-\left(\eta_{\mathfrak{J}}^{+}\right)^{\mathfrak{D}\mathscr{F}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}\right] \\ e^{\imath 2\pi\left[\left(\left(1-\left(1-\left(\mu_{\eta_{\mathfrak{J}}}^{-}\right)^{\mathfrak{D}\mathscr{F}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}, \left(\left(1-\left(1-\left(\mu_{\eta_{\mathfrak{J}}}^{+}\right)^{\mathfrak{D}\mathscr{F}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{F}}}\right]}, \\ \left[\left(1-\left(1-\left(1-\left(1-\left(\zeta_{\mathfrak{J}}^{-}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(1-\left(1-\left(\zeta_{\mathfrak{J}}^{+}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\imath 2\pi\left[\left(1-\left(1-\left(1-\left(1-\left(\mu_{\zeta_{\mathfrak{J}}}^{-}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-\left(1-\left(1-\left(\mu_{\zeta_{\mathfrak{J}}}^{+}\right)^{\mathfrak{D}}\right)^{\mathscr{F}}\right)\right)^{\frac{1}{\mathscr{F}}}\right)^{\frac{1}{\mathfrak{D}}}\right]} \end{pmatrix}$$

$$= \left(\left[\eta_{\mathfrak{J}}^{-}, \eta_{\mathfrak{J}}^{+}\right] e^{\imath 2\pi\left[\mu_{\eta_{\mathfrak{J}}}^{-}, \mu_{\eta_{\mathfrak{J}}}^{+}\right]}, \left[\zeta_{\mathfrak{J}}^{-}, \zeta_{\mathfrak{J}}^{+}\right] e^{\imath 2\pi\left[\mu_{\zeta_{\mathfrak{J}}}^{-}, \mu_{\zeta_{\mathfrak{J}}}^{+}\right]}\right).$$

Theorem 3: (Commutativity) If $\Im'_{\mathfrak{tl}}$ is any permutation of $\Im_{\mathfrak{tl}}$ with the same order of elements, then

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\Im_1, \Im_2, \Im_3, .., \Im_{\mathfrak{p}}\right) = CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\Im'_1, \Im'_2, \Im'_3, .., \Im'_{\mathfrak{p}}\right) \tag{18}$$

Proof:. Consider

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\Im_1, \Im_2, \Im_3, .., \Im_{\mathfrak{p}}\right) = \left(\left(\frac{\oplus_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(\otimes_{\mathfrak{tl}=1}^{\ell}\Im_{\mathfrak{tl}\mathfrak{tl}}\right)}{C_{\mathfrak{p}}^{\ell}}\right)\right)^{\frac{1}{\ell}} = \left(\left(\frac{\oplus_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(\otimes_{\mathfrak{tl}=1}^{\ell}\Im'_{\mathfrak{tl}\mathfrak{tl}}\right)}{C_{\mathfrak{p}}^{\ell}}\right)\right)^{\frac{1}{\ell}} = CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\Im'_1, \Im'_2, \Im'_3, .., \Im'_{\mathfrak{p}}\right)$$

Theorem 4: (Monotonicity) If $\eta^-_{\mathfrak{tl}} \geq \eta'^-_{\mathfrak{tl}}, \zeta^-_{\mathfrak{tl}} \lesssim \zeta'^-_{\mathfrak{tl}}, \mu^-_{\eta_{\mathfrak{tl}}} \geq \mu'^-_{\eta_{\mathfrak{tl}}}, \eta^+_{\mathfrak{tl}} \geq \eta'^+_{\mathfrak{tl}}, \zeta^+_{\mathfrak{tl}} \lesssim \zeta'^+_{\mathfrak{tl}}, \mu^+_{\eta_{\mathfrak{tl}}} \geq \mu'^+_{\eta_{\mathfrak{tl}}}$ and $\mu^-_{\zeta_{\mathfrak{tl}}} \lesssim \mu'^-_{\zeta_{\mathfrak{tl}}}, \mu^+_{\zeta_{\mathfrak{tl}}} \lesssim \mu'^+_{\zeta_{\mathfrak{tl}}}$, then

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\Im_1, \Im_2, \Im_3, .., \Im_{\mathfrak{p}}\right) \geq CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\Im'_1, \Im'_2, \Im'_3, .., \Im'_{\mathfrak{p}}\right) \tag{19}$$

Proof: For $\ell \geq 1$ and $\eta^-_{\mathfrak{tl}} \geq \eta'^-_{\mathfrak{tl}}, \zeta^-_{\mathfrak{tl}} \lesssim \zeta'^-_{\mathfrak{tl}}, \mu^-_{\eta_{\mathfrak{tl}}} \geq \mu'^-_{\eta_{\mathfrak{tl}}}, \eta^+_{\mathfrak{tl}} \geq \eta'^+_{\mathfrak{tl}}, \zeta^+_{\mathfrak{tl}} \lesssim \zeta'^+_{\mathfrak{tl}}, \mu^+_{\eta_{\mathfrak{tl}}} \geq \mu'^+_{\eta_{\mathfrak{tl}}}$ and $\mu^-_{\zeta_{\mathfrak{tl}}} \lesssim \mu'^-_{\zeta_{\mathfrak{tl}}}, \mu^+_{\zeta_{\mathfrak{tl}}} \lesssim \mu'^+_{\zeta_{\mathfrak{tl}}}$. Then,

$$\prod_{\mathfrak{tl}=1}^{\ell}\eta^+_{\mathfrak{tl}_{\mathfrak{tl}}} \geq \prod_{\mathfrak{tl}=1}^{\ell}\eta'^+_{\mathfrak{tl}_{\mathfrak{tl}}} \Rightarrow 1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^+_{\mathfrak{tl}_{\mathfrak{tl}}}\right)^{\mathfrak{D}} \lesssim 1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta'^+_{\mathfrak{tl}_{\mathfrak{tl}}}\right)^{\mathfrak{D}}$$

$$\Rightarrow \prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}} 1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^+_{\mathfrak{tl}_{\mathfrak{tl}}}\right)^{\mathfrak{D}} \lesssim \prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}} 1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta'^+_{\mathfrak{tl}_{\mathfrak{tl}}}\right)^{\mathfrak{D}}$$

$$\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta^+_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}} \geq \left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\eta'^+_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}$$

Similarly, we have

$$\zeta^+_{\mathfrak{tl}} \lesssim \zeta'^+_{\mathfrak{tl}} \Rightarrow 1-\left(\zeta^+_{\mathfrak{tl}}\right)^{\mathfrak{D}} \geq 1-\left(\zeta'^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}$$

$$1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right) \lesssim 1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta'^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)$$

$$\Rightarrow \prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right) \lesssim \prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta'^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)$$

$$\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}} \lesssim \left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta'^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}$$

$$\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}} \geq \left(\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta'^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}$$

$$\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}} \lesssim \left(1-\left(\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta'^+_{\mathfrak{tl}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}$$

Based on the above reservations, we get

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\Im_1, \Im_2, \Im_3, .., \Im_{\mathfrak{p}}\right) \geq CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\Im'_1, \Im'_2, \Im'_3, .., \Im'_{\mathfrak{p}}\right).$$

Theorem 5: (Boundedness). If $\Im^+_{\mathfrak{tl}} = \left(\left[\max_{1\lesssim\mathfrak{tl}\lesssim\ell}\eta^-_{\mathfrak{tl}}, \max_{1\lesssim\mathfrak{tl}\lesssim\ell}\eta^+_{\mathfrak{tl}}\right]e^{\mathfrak{tl}.2\pi\left[\max_{1\lesssim\mathfrak{tl}\lesssim\ell}\mu^-_{\eta_{\mathfrak{tl}}}, \max_{1\lesssim\mathfrak{tl}\lesssim\ell}\mu^+_{\eta_{\mathfrak{tl}}}\right]}, \left[\min_{1\lesssim\mathfrak{tl}\lesssim\ell}\zeta^-_{\mathfrak{tl}}, \min_{1\lesssim\mathfrak{tl}\lesssim\ell}\zeta^+_{\mathfrak{tl}}\right]e^{\mathfrak{tl}.2\pi\left[\min_{1\lesssim\mathfrak{tl}\lesssim\ell}\mu^-_{\zeta_{\mathfrak{tl}}}, \min_{1\lesssim\mathfrak{tl}\lesssim\ell}\mu^+_{\zeta_{\mathfrak{tl}}}\right]}\right)$, and $\Im^+_{\mathfrak{tl}} = \left(\left[\min_{1\lesssim\mathfrak{tl}\lesssim\ell}\eta^-_{\mathfrak{tl}}, \min_{1\lesssim\mathfrak{tl}\lesssim\ell}\eta^+_{\mathfrak{tl}}\right]e^{\mathfrak{tl}.2\pi\left[\min_{1\lesssim\mathfrak{tl}\lesssim\ell}\mu^-_{\eta_{\mathfrak{tl}}}, \min_{1\lesssim\mathfrak{tl}\lesssim\ell}\mu^+_{\eta_{\mathfrak{tl}}}\right]}, \left[\max_{1\lesssim\mathfrak{tl}\lesssim\ell}\zeta^-_{\mathfrak{tl}}, \max_{1\lesssim\mathfrak{tl}\lesssim\ell}\zeta^+_{\mathfrak{tl}}\right]e^{\mathfrak{tl}.2\pi\left[\max_{1\lesssim\mathfrak{tl}\lesssim\ell}\mu^-_{\zeta_{\mathfrak{tl}}}, \max_{1\lesssim\mathfrak{tl}\lesssim\ell}\mu^+_{\zeta_{\mathfrak{tl}}}\right]}\right)$, then

$$\Im^-_{\mathfrak{tl}} \lesssim CI\text{-}Vq\text{-}ROFMSM\left(\Im_1, \Im_2, \Im_3, \ldots, \Im_{\ell}\right) \lesssim \Im^+_{\mathfrak{tl}} \tag{20}$$

Proof: In the presence of Theorem (2, 4), we get

$$\Im^-_{\mathfrak{tl}} = CI\text{-}Vq\text{-}ROFMSM\left(\Im^-_1, \Im^-_2, \Im^-_3, \ldots, \Im^-_{\ell}\right) \lesssim CI\text{-}Vq\text{-}ROFMSM\left(\Im_1, \Im_2, \Im_3, \ldots, \Im_{\ell}\right)$$

$$\Im^+_{\mathfrak{tl}} = CI\text{-}Vq\text{-}ROFMSM\left(\Im^+_1, \Im^+_2, \Im^+_3, \ldots, \Im^+_{\ell}\right) \geq CI\text{-}Vq\text{-}ROFMSM\left(\Im_1, \Im_2, \Im_3, \ldots, \Im_{\ell}\right)$$

Definition 8: The CI-Vq-ROFWMSM operator is simplified by:

$$CI\text{-}Vq\text{-}ROFWMSM^{\ell}\left(\Im_1, \Im_2, \Im_3, .., \Im_{\mathfrak{p}}\right) = \left(\left(\frac{\oplus_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\ell}\lesssim\mathfrak{p}}\left(\otimes_{\mathfrak{tl}=1}^{\ell}\left(\Im_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathbb{W}}_{\mathfrak{tl}\mathfrak{tl}}\right)}{C_{\mathfrak{p}}^{\ell}}\right)\right)^{\frac{1}{\ell}} \tag{21}$$

Theorem 6: In the occurrence of Eq. (21), we get

$$\begin{aligned}&\textit{CI-Vq-ROFWMSM}^{\ell}\left(\mathfrak{I}_1,\mathfrak{I}_2,\mathfrak{I}_3,..,\mathfrak{I}_{\mathfrak{p}}\right)\\&=\left(\begin{array}{c}\left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\eta^{-}_{\mathfrak{tltl}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\ell}_{\mathfrak{p}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\eta^{+}_{\mathfrak{tltl}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\ell}_{\mathfrak{p}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{-}_{\eta_{\mathfrak{tltl}}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\ell}_{\mathfrak{p}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{+}_{\eta_{\mathfrak{tltl}}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\ell}_{\mathfrak{p}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]},\\ \left[\begin{array}{c}\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^{-}_{\mathfrak{tltl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)\right)^{\frac{1}{C^{\ell}_{\mathfrak{p}}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\\ \left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{t}=1}^{\ell}\left(1-\left(\zeta^{+}_{\mathfrak{tltl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)\right)^{\frac{1}{C^{\ell}_{\mathfrak{p}}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\end{array}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\zeta_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)\right)^{\frac{1}{C^{\ell}_{\mathfrak{p}}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\zeta_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)\right)^{\frac{1}{C^{\ell}_{\mathfrak{p}}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right)\end{aligned}$$

Proof: In the availability of Eq. (21), we get

$$\otimes_{\mathfrak{tl}=1}^{\ell}\left(\mathfrak{I}_{\mathfrak{tltl}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}=\left(\begin{array}{c}\left[\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\eta^{-}_{\mathfrak{tltl}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right),\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\eta^{+}_{\mathfrak{tltl}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)\right]e^{\mathfrak{tl}2\pi\left[\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{-}_{\eta_{\mathfrak{tltl}}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right),\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{+}_{\eta_{\mathfrak{tltl}}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)\right]},\\ \left[\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^{-}_{\mathfrak{tltl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^{+}_{\mathfrak{tltl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\zeta_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\zeta_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right)$$

$$\begin{aligned}&\oplus_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{l}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(\otimes_{\mathfrak{tl}=1}^{\ell}\left(\mathfrak{I}_{\mathfrak{tltl}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)\\&=\left(\begin{array}{c}\left[\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\eta^{-}_{\mathfrak{tltl}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\eta^{+}_{\mathfrak{tltl}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{-}_{\eta_{\mathfrak{tltl}}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{+}_{\eta_{\mathfrak{tltl}}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]},\\ \left[\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^{-}_{\mathfrak{tltl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\frac{1}{\mathfrak{D}}},\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\zeta^{+}_{\mathfrak{tltl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim...,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\zeta_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\frac{1}{\mathfrak{D}}},\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_\ell\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\zeta_{\mathfrak{tltl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tltl}}}\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right)\end{aligned}$$

Then, we get

$$\frac{\oplus_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(\otimes_{ꞁ=1}^{\mathfrak{p}}\left(\mathfrak{F}_{ꞁ}\right)^{\mathbb{W}_{ꞁ}}\right)}{C_{\mathfrak{p}}^{\ell}} = \left(\begin{array}{c}\left[\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{ꞁ=1}^{\ell}\left(\eta_{ꞁ}^{-}\right)^{\mathbb{W}_{ꞁ}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{ꞁ=1}^{\ell}\left(\eta_{ꞁ}^{+}\right)^{\mathbb{W}_{ꞁ}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{ꞁ2\pi\left[\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{ꞁ=1}^{\ell}\left(\mu_{\eta_{ꞁ}}^{-}\right)^{\mathbb{W}_{ꞁ}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{ꞁ=1}^{\ell}\left(\mu_{\eta_{ꞁ}}^{+}\right)^{\mathbb{W}_{ꞁ}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{ꞁ=1}^{\ell}\left(1-\left(\zeta_{ꞁ}^{-}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{ꞁ}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}},\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{ꞁ=1}^{\ell}\left(1-\left(\zeta_{ꞁ}^{+}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{ꞁ}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right] \\ e^{ꞁ2\pi\left[\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{ꞁ=1}^{\ell}\left(1-\left(\mu_{\zeta_{ꞁ}}^{-}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{ꞁ}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}},\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{ꞁ=1}^{\ell}\left(1-\left(\mu_{\zeta_{ꞁ}}^{+}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{ꞁ}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right]}\end{array}\right)$$

$$CI\text{-}Vq\text{-}ROFMSM^{\ell}\left(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,..,\mathfrak{F}_{\mathfrak{p}}\right) = \left(\begin{array}{c}\left[\left(\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{ꞁ=1}^{\ell}\left(\eta_{ꞁ}^{-}\right)^{\mathbb{W}_{ꞁ}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{ꞁ=1}^{\ell}\left(\eta_{ꞁ}^{+}\right)^{\mathbb{W}_{ꞁ}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right] \\ e^{ꞁ2\pi\left[\left(\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{ꞁ=1}^{\ell}\left(\mu_{\eta_{ꞁ}}^{-}\right)^{\mathbb{W}_{ꞁ}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{ꞁ=1}^{\ell}\left(\mu_{\eta_{ꞁ}}^{+}\right)^{\mathbb{W}_{ꞁ}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]}, \\ \left[\begin{array}{c}\left(1-\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{ꞁ=1}^{\ell}\left(1-\left(\zeta_{ꞁ}^{-}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{ꞁ}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}, \\ \left(1-\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{ꞁ=1}^{\ell}\left(1-\left(\zeta_{ꞁ}^{+}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{ꞁ}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\end{array}\right] \\ e^{ꞁ2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{ꞁ=1}^{\ell}\left(1-\left(\mu_{\zeta_{ꞁ}}^{-}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{ꞁ}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{ꞁ=1}^{\ell}\left(1-\left(\mu_{\zeta_{ꞁ}}^{+}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{ꞁ}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right)$$

Definition 9: The CI-Vq-ROFDMSM operator is diagnosed by:

$$CI\text{-}Vq\text{-}ROFDMSM^{\ell}\left(\mathfrak{F}_1,\mathfrak{F}_2,\mathfrak{F}_3,..,\mathfrak{F}_{\mathfrak{p}}\right) = \frac{1}{\ell}\left(\otimes_{1\lesssim ꞁ_1\lesssim ꞁ_2\lesssim ..,\lesssim ꞁ_{\ell}\lesssim \mathfrak{p}}\left(\oplus_{ꞁ=1}^{\ell}\mathfrak{F}_{ꞁ}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right) \quad (22)$$

Theorem 7: The CI-Vq-ROFDMSM operator is diagnosed by

$$
\begin{aligned}
&\textit{CI-Vq-ROFDMSM}^{\ell}\left(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3,..,\mathfrak{J}_{\mathfrak{p}}\right)\\
&=\begin{pmatrix}
\left[\begin{array}{c}
\left(1-\left(1-\left(\prod_{1\lesssim \text{tl}_1\lesssim \text{tl}_2\lesssim ..,\lesssim \text{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\eta_{\text{tltl}}^{-}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\\
\left(1-\left(1-\left(\prod_{1\lesssim \text{tl}_1\lesssim \text{tl}_2\lesssim ..,\lesssim \text{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\eta_{\text{tltl}}^{+}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}
\end{array}\right]\\
e^{\text{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim \text{tl}_1\lesssim \text{tl}_2\lesssim ..,\lesssim \text{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\mu_{\eta_{\text{tltl}}}^{-}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{1\lesssim \text{tl}_1\lesssim \text{tl}_2\lesssim ..,\lesssim \text{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\mu_{\eta_{\text{tltl}}}^{+}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right]},\\
\left[\left(\left(1-\left(\prod_{1\lesssim \text{tl}_1\lesssim \text{tl}_2\lesssim ..,\lesssim \text{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{\text{tl}=1}^{\ell}\zeta_{\text{tltl}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim \text{tl}_1\lesssim \text{tl}_2\lesssim ..,\lesssim \text{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{\text{tl}=1}^{\ell}\zeta_{\text{tltl}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]\\
e^{\text{tl}2\pi\left[\left(\left(1-\left(\prod_{1\lesssim \text{tl}_1\lesssim \text{tl}_2\lesssim ..,\lesssim \text{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{\text{tl}=1}^{\ell}\mu_{\zeta_{\text{tltl}}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim \text{tl}_1\lesssim \text{tl}_2\lesssim ..,\lesssim \text{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{\text{tl}=1}^{\ell}\mu_{\zeta_{\text{tltl}}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]}
\end{pmatrix}
\end{aligned}
\tag{22}
$$

Proof: In the availability of Eq. (22), we have

$$
\oplus_{\text{tl}=1}^{\ell}\mathfrak{J}_{\text{tltl}}=\begin{pmatrix}
\left[\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\eta_{\text{tltl}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\eta_{\text{tltl}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]e^{\text{tl}2\pi\left[\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\mu_{\eta_{\text{tltl}}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\mu_{\eta_{\text{tltl}}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]},\\
\left[\prod_{\text{tl}=1}^{\ell}\zeta_{\text{tltl}}^{-},\prod_{\text{tl}=1}^{\ell}\zeta_{\text{tltl}}^{+}\right]e^{\text{tl}2\pi\left[\prod_{\text{tl}=1}^{\ell}\mu_{\zeta_{\text{tltl}}}^{-},\prod_{\text{tl}=1}^{\ell}\mu_{\zeta_{\text{tltl}}}^{+}\right]}
\end{pmatrix}
$$

$$
\left(\oplus_{\text{tl}=1}^{\ell}\mathfrak{J}_{\text{tltl}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}=\begin{pmatrix}
\left[\left(\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\eta_{\text{tltl}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}},\left(\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\eta_{\text{tltl}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right]\\
e^{\text{tl}2\pi\left[\left(\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\mu_{\eta_{\text{tltl}}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}},\left(\left(1-\prod_{\text{tl}=1}^{\ell}\left(1-\left(\mu_{\eta_{\text{tltl}}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right]},\\
\left[\left(1-\left(1-\left(\prod_{\text{tl}=1}^{\ell}\zeta_{\text{tltl}}^{-}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{\text{tl}=1}^{\ell}\zeta_{\text{tltl}}^{+}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right]\\
e^{\text{tl}2\pi\left[\left(1-\left(1-\left(\prod_{\text{tl}=1}^{\ell}\mu_{\zeta_{\text{tltl}}}^{-}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{\text{tl}=1}^{\ell}\mu_{\zeta_{\text{tltl}}}^{+}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right]}
\end{pmatrix}
$$

Then, we have

$$\otimes_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(\oplus_{\mathfrak{tl}=1}^{\mathscr{f}}\Im_{\mathfrak{tl}\mathfrak{tl}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}$$

$$=\begin{pmatrix} \left[\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\eta_{\mathfrak{tl}\mathfrak{tl}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\eta_{\mathfrak{tl}\mathfrak{tl}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\mathfrak{tl}2\pi\left[\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\zeta_{\mathfrak{tl}\mathfrak{tl}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\zeta_{\mathfrak{tl}\mathfrak{tl}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]} \end{pmatrix}$$

Therefore,

$$CI\text{-}Vq\text{-}ROFMSM^{\mathscr{f}}\left(\Im_1,\Im_2,\Im_3,..,\Im_{\mathfrak{p}}\right)$$

$$=\begin{pmatrix} \left[\begin{array}{c}\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\eta_{\mathfrak{tl}\mathfrak{tl}}^{-}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathscr{f}}}\right)^{\frac{1}{\mathfrak{D}}}, \\ \left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\eta_{\mathfrak{tl}\mathfrak{tl}}^{+}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathscr{f}}}\right)^{\frac{1}{\mathfrak{D}}}\end{array}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^{-}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathscr{f}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^{+}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{\mathfrak{p}^{\mathscr{f}}}}\right)^{\frac{1}{\mathscr{f}}}\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\zeta_{\mathfrak{tl}\mathfrak{tl}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{f}}},\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\zeta_{\mathfrak{tl}\mathfrak{tl}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{f}}}\right] \\ e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{f}}},\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{f}}}\right]} \end{pmatrix}$$

Theorem 8: (Idempotency) If $\Im_{\mathfrak{tl}}=\left(\left[\eta_{\mathfrak{tl}}^{-},\eta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}2\pi\left[\mu_{\eta_{\mathfrak{tl}}}^{-},\mu_{\eta_{\mathfrak{tl}}}^{+}\right]},\left[\zeta_{\mathfrak{tl}}^{-},\zeta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}2\pi\left[\mu_{\zeta_{\mathfrak{tl}}}^{-},\mu_{\zeta_{\mathfrak{tl}}}^{+}\right]}\right)=\Im$, $\mathfrak{tl}=1,2,3,..,\mathfrak{p}$, then

$$CI\text{-}Vq\text{-}ROFDMSM^{\mathscr{f}}\left(\Im_1,\Im_2,\Im_3,..,\Im_{\mathfrak{p}}\right)=\Im \tag{23}$$

Proof:. Straightforward.

Theorem 9: (Commutativity) If $\Im_{\mathfrak{tl}}'$ is any permutation of $\Im_{\mathfrak{tl}}$ with the same order of elements, then

$$CI\text{-}Vq\text{-}ROFDMSM^{\mathscr{f}}\left(\Im_1,\Im_2,\Im_3,..,\Im_{\mathfrak{p}}\right) = CI\text{-}Vq\text{-}ROFDMSM^{\mathscr{f}}\left(\Im_1',\Im_2',\Im_3',..,\Im_{\mathfrak{p}}'\right)(24) \tag{24}$$

Proof:. Straightforward.

Theorem 10: (Monotonicity) If $\eta_{\mathfrak{tl}}^{-}\geq\eta_{\mathfrak{tl}}'^{-}$, $\zeta_{\mathfrak{tl}}^{-}\lesssim\zeta_{\mathfrak{tl}}'^{-},\mu_{\eta_{\mathfrak{tl}}}^{-}\geq\mu_{\eta_{\mathfrak{tl}}}'^{-},\eta_{\mathfrak{tl}}^{+}\geq\eta_{\mathfrak{tl}}'^{+},\zeta_{\mathfrak{tl}}^{+}\lesssim\zeta_{\mathfrak{tl}}'^{+},\mu_{\eta_{\mathfrak{tl}}}^{+}\geq\mu_{\eta_{\mathfrak{tl}}}'^{+}$ and $\mu_{\zeta_{\mathfrak{tl}}}^{-}\lesssim\mu_{\zeta_{\mathfrak{tl}}}'^{-}$, $\mu_{\zeta_{\mathfrak{tl}}}^{+}\lesssim\mu_{\zeta_{\mathfrak{tl}}}'^{+}$, then

$$CI\text{-}Vq\text{-}ROFDMSM^{\mathscr{f}}\left(\Im_1,\Im_2,\Im_3,..,\Im_{\mathfrak{p}}\right) \geq CI\text{-}Vq\text{-}ROFDMM\left(\Im_1',\Im_2',\Im_3',..,\Im_{\mathfrak{p}}'\right) \tag{25}$$

Proof: Straightforward.

Theorem 11: (Boundedness). If $\Im_{\mathfrak{tl}}^{+}=\left(\left[1\lesssim\mathfrak{tl}\lesssim\mathscr{f}max\eta_{\mathfrak{tl}}^{-},1\lesssim\mathfrak{tl}\lesssim\mathscr{f}max\eta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}.2\pi\left[1\lesssim\mathfrak{tl}\lesssim\mathscr{f}max\mu_{\eta_{\mathfrak{tl}}}^{-},1\lesssim\mathfrak{tl}\lesssim\mathscr{f}max\mu_{\eta_{\mathfrak{tl}}}^{+}\right]},\left[1\lesssim\mathfrak{tl}\lesssim\mathscr{f}min\zeta_{\mathfrak{tl}}^{-},1\lesssim\mathfrak{tl}\lesssim\mathscr{f}min\zeta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}.2\pi\left[1\lesssim\mathfrak{tl}\lesssim\mathscr{f}min\mu_{\zeta_{\mathfrak{tl}}}^{-},1\lesssim\mathfrak{tl}\lesssim\mathscr{f}min\mu_{\zeta_{\mathfrak{tl}}}^{+}\right]}\right)$ and $\Im_{\mathfrak{tl}}^{-}=\left(\left[1\lesssim\mathfrak{tl}\lesssim\mathscr{f}min\eta_{\mathfrak{tl}}^{-},1\lesssim\mathfrak{tl}\lesssim\mathscr{f}min\eta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}.2\pi\left[1\lesssim\mathfrak{tl}\lesssim\mathscr{f}min\mu_{\eta_{\mathfrak{tl}}}^{-},1\lesssim\mathfrak{tl}\lesssim\mathscr{f}min\mu_{\eta_{\mathfrak{tl}}}^{+}\right]},\left[1\lesssim\mathfrak{tl}\lesssim\mathscr{f}max\zeta_{\mathfrak{tl}}^{-},1\lesssim\mathfrak{tl}\lesssim\mathscr{f}max\zeta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}.2\pi\left[1\lesssim\mathfrak{tl}\lesssim\mathscr{f}max\mu_{\zeta_{\mathfrak{tl}}}^{-},1\lesssim\mathfrak{tl}\lesssim\mathscr{f}max\mu_{\zeta_{\mathfrak{tl}}}^{+}\right]}\right)$, then

$$\Im_{\mathfrak{tl}}^{-}\lesssim CI\text{-}Vq\text{-}ROFDMSM\left(\Im_1,\Im_2,\Im_3,\ldots,\Im_{\mathscr{f}}\right)\lesssim\Im_{\mathfrak{tl}}^{+} \tag{26}$$

Proof: Straightforward.

Definition 14: The Cq-ROFWDMSM operator is simplified by:

$$CI\text{-}Vq\text{-}ROFWDMSM^{\mathscr{f}}\left(\Im_1,\Im_2,\Im_3,..,\Im_{\mathfrak{p}}\right) = \frac{1}{\mathscr{f}}\left(\otimes_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(\oplus_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}\otimes\Im_{\mathfrak{tl}\mathfrak{tl}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right) \tag{27}$$

Theorem 12: In the availability of Eq. (27), we have

$$
\begin{aligned}
&CI\text{-}Vq\text{-}ROFWDMSM^{\ell}\left(\mathfrak{J}_1,\mathfrak{J}_2,\mathfrak{J}_3,..,\mathfrak{J}_{\mathfrak{p}}\right)\\
&=\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1-\left(1-\left(\prod_{1\lesssim \mathfrak{tl}_1\lesssim \mathfrak{tl}_2\lesssim ..,\lesssim \mathfrak{tl}_{\ell}\lesssim}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\eta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\\
\left(1-\left(1-\left(\prod_{1\lesssim \mathfrak{tl}_1\lesssim \mathfrak{tl}_2\lesssim ..,\lesssim \mathfrak{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\eta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}
\end{array}\right]\\
e^{\mathfrak{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim \mathfrak{tl}_1\lesssim \mathfrak{tl}_2\lesssim ...\lesssim \mathfrak{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{1\lesssim \mathfrak{tl}_1\lesssim \mathfrak{tl}_2\lesssim ...\lesssim \mathfrak{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\ell}}\right)^{\frac{1}{\mathfrak{D}}}\right]},\\
\left[\begin{array}{c}
\left(\left(1-\left(\prod_{1\lesssim \mathfrak{tl}_1\lesssim \mathfrak{tl}_2\lesssim ..,\lesssim \mathfrak{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\zeta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\\
\left(\left(1-\left(\prod_{1\lesssim \mathfrak{tl}_1\lesssim \mathfrak{tl}_2\lesssim ..,\lesssim \mathfrak{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\zeta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}
\end{array}\right]\\
e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(\prod_{1\lesssim \mathfrak{tl}_1\lesssim \mathfrak{tl}_2\lesssim ...\lesssim \mathfrak{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{-}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}},\left(\left(1-\left(\prod_{1\lesssim \mathfrak{tl}_1\lesssim \mathfrak{tl}_2\lesssim ...\lesssim \mathfrak{tl}_{\ell}\lesssim \mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{+}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\ell}}\right]}
\end{array}\right)
\end{aligned}
\tag{28}
$$

Proof: In the availability of Eq. (28), we have

$$
\oplus_{\mathfrak{tl}=1}^{\ell}\left(\mathfrak{t}_{\mathfrak{tl}}\otimes\mathfrak{J}_{\mathfrak{tl}\mathfrak{tl}}\right)=\left(\begin{array}{c}
\left[\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\eta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\eta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right]\\
e^{\mathfrak{tl}2\pi\left[\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right]},\\
\left[\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\zeta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right),\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\zeta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)\right]e^{\mathfrak{tl}2\pi\left[\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{-}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right),\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{+}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)\right]}
\end{array}\right)
$$

$$
\left(\oplus_{\mathfrak{tl}=1}^{\ell}\left(\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}\otimes\mathfrak{J}_{\mathfrak{tl}\mathfrak{tl}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}=\left(\begin{array}{c}
\left[\left(\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\eta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}},\left(\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\eta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right]\\
e^{\mathfrak{tl}2\pi\left[\left(\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{-}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}},\left(\left(1-\prod_{\mathfrak{tl}=1}^{\ell}\left(1-\left(\mu^{+}_{\eta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right]},\\
\left[\left(1-\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\zeta^{-}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\zeta^{+}_{\mathfrak{tl}\mathfrak{tl}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right]\\
e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{-}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right),\left(\left(1-\left(1-\left(\prod_{\mathfrak{tl}=1}^{\ell}\left(\mu^{+}_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\ell}}}\right)^{\frac{1}{\mathfrak{D}}}\right)\right]}
\end{array}\right)
$$

Then, we get

$$\otimes_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(\oplus_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}\otimes\mathfrak{I}_{\mathfrak{tl}\mathfrak{tl}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}$$

$$=\begin{pmatrix}\left[\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\eta_{\mathfrak{tl}\mathfrak{tl}}^{-}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}},\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\eta_{\mathfrak{tl}\mathfrak{tl}}^{+}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^{-}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}},\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^{+}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right]},\\ \left[\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\zeta_{\mathfrak{tl}\mathfrak{tl}}^{-}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\zeta_{\mathfrak{tl}\mathfrak{tl}}^{+}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^{-}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right),\left(\left(1-\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^{+}\right)^{\mathbb{W}_{\mathfrak{tl}\mathfrak{tl}}}\right)^{\mathfrak{D}}\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)\right]}\end{pmatrix}$$

$$C\text{-}Vq\text{-}ROFWDMSM^{\mathscr{f}}\left(\mathfrak{I}_1,\mathfrak{I}_2,\mathfrak{I}_3,..,\mathfrak{I}_{\mathfrak{p}}\right)$$

$$=\begin{pmatrix}\left[\begin{matrix}\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{t}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\eta_{\mathfrak{tl}\mathfrak{tl}}^{-}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathscr{f}}}\right)^{\frac{1}{\mathfrak{D}}},\\ \left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\eta_{\mathfrak{tl}\mathfrak{tl}}^{+}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathscr{f}}}\right)^{\frac{1}{\mathfrak{D}}}\end{matrix}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^{-}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathscr{f}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(1-\left(\mu_{\eta_{\mathfrak{tl}\mathfrak{tl}}}^{+}\right)^{\mathfrak{D}}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathscr{f}}}\right)^{\frac{1}{\mathfrak{D}}}\right]},\\ \left[\begin{matrix}\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\zeta_{\mathfrak{tl}\mathfrak{tl}}^{-}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{f}}},\\ \left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\zeta_{\mathfrak{tl}\mathfrak{tl}}^{+}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{f}}}\end{matrix}\right]\\ e^{\mathfrak{tl}2\pi\left[\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^{-}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{f}}},\left(\left(1-\left(\prod_{1\lesssim\mathfrak{tl}_1\lesssim\mathfrak{tl}_2\lesssim..,\lesssim\mathfrak{tl}_{\mathscr{f}}\lesssim\mathfrak{p}}\left(1-\left(\prod_{\mathfrak{tl}=1}^{\mathscr{f}}\left(\mu_{\zeta_{\mathfrak{tl}\mathfrak{tl}}}^{+}\right)^{\mathbb{W}_{\mathfrak{tl}_{\mathfrak{tl}}}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_{\mathfrak{p}}^{\mathscr{f}}}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{\mathscr{f}}}\right]}\end{pmatrix}$$

4. Application (MADM Model for CI-Vq-ROFNs)

We invent a MADM tool for the availability of CI-Vq-ROFMSM, CI-Vq-ROFWMSM, CI-Vq-ROFDMSM, and CI-Vq-ROFWDMSM operators. The data used in this section are taken from Rong et al. (2020).

For evaluating a decision-making tool, called the MADM technique, we assume a mathematical term $\mathfrak{R}\mathfrak{f}_1,\mathfrak{R}\mathfrak{f}_2,\ldots,\mathfrak{R}\mathfrak{f}_m$, show the alternatives and the mathematical terms $G_1,G_2,\ldots,G_{\mathfrak{p}}$, stated the attributes with various weight vectors $\mathbb{W}=\{\mathbb{W}_1,\mathbb{W}_2,\ldots,\mathbb{W}_{\mathfrak{p}}\}^T$, and hold the rule $\mathbb{W}_{\mathfrak{tl}}\in[0,1]$, and $\sum_{\mathfrak{tl}=1}^{\mathfrak{p}}\mathbb{W}_{\mathfrak{tl}}=1$. To resolve the above assessments in the consideration of exposed approaches, we assume a Matrix whose defined value is CI-Vq-ROFNs, i.e., $\left(\left[\eta_{\mathfrak{tl}}^{-},\eta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}2\pi\left[\mu_{\eta_{\mathfrak{tl}}}^{-},\mu_{\eta_{\mathfrak{tl}}}^{+}\right]},\left[\zeta_{\mathfrak{tl}}^{-},\zeta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}2\pi\left[\mu_{\zeta_{\mathfrak{tl}}}^{-},\mu_{\zeta_{\mathfrak{tl}}}^{+}\right]}\right)$, by suggesting the values: $\eta'_{\mathfrak{R}\mathfrak{f}}=\left[\eta_{\mathfrak{tl}}^{-},\eta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}2\pi\left[\mu_{\eta_{\mathfrak{tl}}}^{-},\mu_{\eta_{\mathfrak{tl}}}^{+}\right]}$ and $\zeta'_{\mathfrak{R}\mathfrak{f}}=\left[\zeta_{\mathfrak{tl}}^{-},\zeta_{\mathfrak{tl}}^{+}\right]e^{\mathfrak{tl}2\pi\left[\mu_{\zeta_{\mathfrak{tl}}}^{-},\mu_{\zeta_{\mathfrak{tl}}}^{+}\right]}$, in the availability of several rules: $0\lesssim\eta'_{\mathfrak{R}\mathfrak{f}}{}^{+\mathfrak{D}}(u)+\zeta'_{\mathfrak{R}\mathfrak{f}}{}^{+\mathfrak{D}}(u)\lesssim 1$ and $0\lesssim W_{T_\tau}^{+\mathfrak{D}}(u)+W_{F_\tau}^{+\mathfrak{D}}(u)\lesssim 1$. Finally, with the help of a new algorithm, we try to evaluate the above types of problems whose steps are diagnosed here:

Step 1: Arrange the family of CI-Vq-ROFNs.
Step 2: Compute the CI-Vq-ROFNs with the help of CI-Vq-ROFWMSM and CI-Vq-ROFWDMSM operators.
Step 3: Compute the score values.
Step 4: Compute the ranking values and try to find the best one.

4.1. Illustrative example

We give various hypostatical sorts of information to evaluate the potential resolution of emergency technology commercialization (ETC) based on CI-Vq-ROF information to describe the invented approaches in the study. The suggested data are taken from Rong et al. (2020).

Consider four possible ETC enterprises with a mathematical form $\mathfrak{R}\mathfrak{f}_1, \mathfrak{R}\mathfrak{f}_2, \mathfrak{R}\mathfrak{f}_3, \mathfrak{R}\mathfrak{f}_4$, and for this expert, they suggest the four attributes in the form G_1, G_2, G_3, G_4 with expressions as discussed as follows:

1. G_1: Technical Advancement.
2. G_2: Political Market.
3. G_3: Financial Factors.
4. G_4: Science and Technology.

To evaluate the above problem, we considered four weight vectors for four attributes such as $w = \{0.4, 0.3, 0.2, 0.1\}^T$. Finally, with the help of a new algorithm, we try to evaluate the above types of problems whose steps are diagnosed as follows:

Step 1: Arrange the family of CI-Vq-ROFNs, as stated in Table 1.

Table 1
The expressions of CI-Vq-ROFNs

Data representation	G_1	G_2
$\mathfrak{R}\mathfrak{f}_1$	$([0.1, 0.2]e^{\mathrm{tI}2\pi[0.2,0.4]}, [0.3, 0.4]e^{\mathrm{tI}2\pi[0.3,0.4]})$	$([0.11, 0.21]e^{\mathrm{tI}2\pi[0.21,0.41]}, [0.31, 0.41]e^{\mathrm{tI}2\pi[0.31,0.41]})$
$\mathfrak{R}\mathfrak{f}_2$	$([0.2, 0.4]e^{\mathrm{tI}2\pi[0.3,0.5]}, [0.1, 0.3]e^{\mathrm{tI}2\pi[0.2,0.3]})$	$([0.21, 0.41]e^{\mathrm{tI}2\pi[0.31,0.51]}, [0.11, 0.31]e^{\mathrm{tI}2\pi[0.21,0.31]})$
$\mathfrak{R}\mathfrak{f}_3$	$([0.1, 0.4]e^{\mathrm{tI}2\pi[0.3,0.4]}, [0.4, 0.5]e^{\mathrm{tI}2\pi[0.4,0.5]})$	$([0.11, 0.41]e^{\mathrm{tI}2\pi[0.31,0.41]}, [0.41, 0.51]e^{\mathrm{tI}2\pi[0.41,0.51]})$
$\mathfrak{R}\mathfrak{f}_4$	$([0.2, 0.3]e^{\mathrm{tI}2\pi[0.1,0.2]}, [0.1, 0.3]e^{\mathrm{tI}2\pi[0.3,0.4]})$	$([0.21, 0.31]e^{\mathrm{tI}2\pi[0.11,0.21]}, [0.11, 0.31]e^{\mathrm{tI}2\pi[0.31,0.41]})$
Data representation	G_3	G_4
$\mathfrak{R}\mathfrak{f}_1$	$([0.12, 0.22]e^{\mathrm{tI}2\pi[0.22,0.42]}, [0.32, 0.42]e^{\mathrm{tI}2\pi[0.32,0.42]})$	$([0.13, 0.23]e^{\mathrm{tI}2\pi[0.23,0.43]}, [0.33, 0.43]e^{\mathrm{tI}2\pi[0.33,0.43]})$
$\mathfrak{R}\mathfrak{f}_2$	$([0.22, 0.42]e^{\mathrm{tI}2\pi[0.32,0.52]}, [0.12, 0.32]e^{\mathrm{tI}2\pi[0.22,0.32]})$	$([0.23, 0.43]e^{\mathrm{tI}2\pi[0.33,0.53]}, [0.13, 0.33]e^{\mathrm{tI}2\pi[0.23,0.33]})$
$\mathfrak{R}\mathfrak{f}_3$	$([0.12, 0.42]e^{\mathrm{tI}2\pi[0.32,0.42]}, [0.42, 0.52]e^{\mathrm{tI}2\pi[0.42,0.52]})$	$([0.13, 0.43]e^{\mathrm{tI}2\pi[0.33,0.43]}, [0.43, 0.53]e^{\mathrm{tI}2\pi[0.43,0.53]})$
$\mathfrak{f}_4$	$([0.22, 0.32]e^{\mathrm{tI}2\pi[0.12,0.22]}, [0.12, 0.32]e^{\mathrm{tI}2\pi[0.32,0.42]})$	$([0.23, 0.33]e^{\mathrm{tI}2\pi[0.13,0.23]}, [0.13, 0.33]e^{\mathrm{tI}2\pi[0.33,0.43]})$

Step 2: Compute the CI-Vq-ROFNs with the help of the CI-Vq-ROFWMSM operator, as described in Table 2.

Table 2
The expressions of aggregated values using the weighted MSM

Data representation	G_1
$\mathfrak{R}\mathfrak{f}_1$	$([0.7863, 0.8257]e^{\mathrm{tI}2\pi[0.8257,0.8544]}, [0.071, 0.0896]e^{\mathrm{tI}2\pi[0.071,0.0896]})$
$\mathfrak{R}\mathfrak{f}_2$	$([0.8257, 0.8781]e^{\mathrm{tI}2\pi[0.8544,0.899]}, [0.0334, 0.071]e^{\mathrm{tI}2\pi[0.0526,0.071]})$
$\mathfrak{R}\mathfrak{f}_3$	$([0.7863, 0.8781]e^{\mathrm{tI}2\pi[0.8544,0.8781]}, [0.0896, 0.1095]e^{\mathrm{tI}2\pi[0.0896,0.1095]})$
$\mathfrak{R}\mathfrak{f}_4$	$([0.8257, 0.8544]e^{\mathrm{tI}2\pi[0.7863,0.8257]}, [0.0334, 0.071]e^{\mathrm{tI}2\pi[0.071,0.0896]})$

Similarly, again compute the CI-Vq-ROFNs with the help of the CI-Vq-ROFWDMSM operator, as described in Table 3.

Table 3
The expressions of aggregated values using the weighted dual MSM

Data representation	G_1
$\mathfrak{R}\mathfrak{f}_1$	$([0.0334, 0.0526]e^{\mathrm{tI}2\pi[0.0526,0.071]}, [0.8544, 0.8781]e^{\mathrm{I}2\pi[0.8544,0.8781]})$
$\mathfrak{R}\mathfrak{f}_2$	$([0.0526, 0.0896]e^{\mathrm{tI}2\pi[0.071,0.1095]}, [0.7863, 0.8544]e^{\mathrm{tI}2\pi[0.8257,0.8544]})$
$\mathfrak{R}\mathfrak{f}_3$	$([0.0334, 0.0896]e^{\mathrm{tI}2\pi[0.071,0.0896]}, [0.8781, 0.899]e^{\mathrm{tI}2\pi[0.8781,0.899]})$
$\mathfrak{R}\mathfrak{f}_4$	$([0.0526, 0.071]e^{\mathrm{tI}2\pi[0.0334,0.0526]}, [0.7863, 0.8544]e^{\mathrm{tI}2\pi[0.8544,0.8781]})$

Step 3: Compute the score values, as described in Table 4.

Table 4
The ranking values of the aggregated values

Data representation	CI-Vq-ROFWMSM operator	CI-Vq-ROFWDMSM operator
$\mathfrak{R}\mathfrak{f}_1$	0.7427	0.8139
$\mathfrak{R}\mathfrak{f}_2$	0.8073	0.7495
$\mathfrak{R}\mathfrak{f}_3$	0.7497	0.8176
$\mathfrak{R}\mathfrak{f}_4$	0.7568	0.7909

Step 4: Compute the ranking values and try to find the best one, such that
For CI-Vq-ROFWMSM operator:

$$\mathfrak{R}\mathfrak{f}_2 > \mathfrak{R}\mathfrak{f}_4 > \mathfrak{R}\mathfrak{f}_3 > \mathfrak{R}\mathfrak{f}_1$$

The best option is $\mathfrak{R}\mathfrak{f}_2$.
For CI-Vq-ROFWDMSM operator:

$$\mathfrak{R}\mathfrak{f}_3 > \mathfrak{R}\mathfrak{f}_1 > \mathfrak{R}\mathfrak{f}_4 > \mathfrak{R}\mathfrak{f}_2$$

The best option is $\mathfrak{R}\mathfrak{f}_3$.

5. Comparative Analysis

A sensitive analysis is one of the fundamental parts of well-known and qualitative manuscripts. In this work, we compare the exposed approaches with various prevailing operators, such as Hamy mean operators for I-VIFS (Wu et al., 2019), Dombi Hamy mean operators for I-VIFS (Wu et al., 2018), MSM for q-ROFSs (Wei et al., 2019), and aggregation operators for CI-Vq-ROFSs (Garg et al., 2021). In the availability of data in Table 1, Table 5 includes the sensitivity analysis of the exposed and existing drawbacks.

Table 5
The comparative analysis

Methods	Score values	Ranking values
Wu et al. (2019)	Failed to find the score values	Failed to find the ranking values
Wu et al. (2018)	Failed to find the score values	Failed to find the ranking values
Wei et al. (2019)	Failed to find the score values	Failed to find the ranking values
Garg et al. (2021)	−0.15, 0.1253, −0.15, −0.075	$\Re\mathfrak{f}_2 > \Re\mathfrak{f}_4 > \Re\mathfrak{f}_3 = \Re\mathfrak{f}_1$
CI-Vq-ROFWMSM operator	0.7427, 0.8073, 0.7497, 0.7568	$\Re\mathfrak{f}_2 > \Re\mathfrak{f}_4 > \Re\mathfrak{f}_3 > \Re\mathfrak{f}_1$
CI-Vq-ROFWDMSM operator	0.8139, 0.7495, 0.8176, 0.7909	$\Re\mathfrak{f}_3 > \Re\mathfrak{f}_1 > \Re\mathfrak{f}_4 > \Re\mathfrak{f}_2$

To more clearly explain the information in Table 5, it is noticed that the information given in Wu et al. (2019) has been failed to evaluate the information given in Table 1, because Table 1 contains the CI-Vq-ROF information and prevailing operators are not able to handle it. The main reason is that the operators defined in Wu et al. (2019) are based on I-VIFS which is the specific part of the invented work. Further, it is noticed that the information given in Wu et al. (2018) has been failed to evaluate the information given in Table 1, because Table 1 contains the CI-Vq-ROF information and prevailing operators are not able to handle it. The main reason is that the operators defined in Wu et al. (2018) are based on I-VIFS which is the specific part of the invented work. And similarly, it is noticed that the information given in Wei et al. (2019) has been failed to evaluate the information given in Table 1, because Table 1 contains the CI-Vq-ROF information and prevailing operators are not able to handle it. The main reason is that the operators defined in Wei et al. (2019) are based on q-ROFS which is the specific part of the invented work. But instead of these theories, the aggregation operators in Garg et al. (2021) are evaluated by using the CI-Vq-ROFSs, which can easily evaluate the presented information given in Table 1. The main influence of the proposed work is that the existing information in Wu et al. (2018, 2019) and Wei et al. (2019) is the specific case of the invented work, but the theory given in Garg et al. (2021) is not the special cases of the diagnosed work, but the operators defined in Garg et al. (2021) are the special cases of the proposed MSM operator, because we defined the MSM operators using the invented CI-Vq-ROFSs. Finally, using the invented theory, we get two different results by using three different tools and the best option is $\Re\mathfrak{f}_2$ using the Garg et al. (2021), and CI-Vq-ROFWMSM operator, get as the best option $\Re\mathfrak{f}_3$ using the CI-Vq-ROFWDMSM operator. Therefore, the exposed approaches are massive dominant and feasible compared to existing drawbacks.

6. Conclusion

Main analyzations are described as follows:

1. To enhance the superiority of the research works, in this scenario, we used the informative idea of the CIVq-ROF setting and took a valuable tool of MSM to present the CIVq-ROFMSM, CIVq-ROFWMSM, CIVq-ROFDMSM, and CIVq-ROFWDMSM operators.
2. To verify the supremacy of the invented works for the different values of parameters, several specific cases are also explored.
3. Finally, with the help of MADM skills, we identified a beneficial optimal in the presence of the source of descriptions in the form of invented operators using the decision-making process.
4. Comparison of the invented approaches with many existing scenarios is also simplified at the end of this analysis, which shows the dominancy and competency of the diagnosed approaches.

In the occurrence of the existing spherical FSs, Ali et al. (2020) T-complex spherical FSs, various important techniques are diagnosed to evaluate and utilize various aggregation operators, new methods, and many techniques to enhance the worth of the prevailing works.

Data Availability

The data used to support the findings of this study can be used by anyone without prior permission of the authors by just citing this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

Ethics Declaration Statement

The authors declare that this is their original work and it is neither submitted nor under consideration in any other journal simultaneously.

Informed Consent

All authors agreed and informed to submit this paper in the journal “Soft Computing” for possible publication.

Authors’ Contributions

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