Received: 3 January 2022 | Revised: 11 February 2022 | Accepted: 13 February 2022 | Published online: 17 February 2022

Journal of Computational and Cognitive Engineering
2022, Vol. 1(1) 42–46
DOI: 10.47852/bonviewJCCE19919205514

BON VIEW PUBLISHING

RESEARCH ARTICLE

# A New Study Based on Fuzzy Bi-Γ-Ideals in Ordered-Γ-Semigroups

**G. Muhiuddin[1,*], A. Mahboob[2] and Mohamed E. A. Elnair[1,3]**

[1]*Department of Mathematics, University of Tabuk, Saudi Arabia*

[2]*Department of Mathematics, Madanapalle Institute of Technology & Science, India*

[3]*Department of Mathematics and Physics, Gezira University, Sudan*

**Abstract:** In this paper, using the notion of $(k^*, q)$-quasi-coincident of an ordered fuzzy point with a fuzzy set of the support, the concept of $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-Γ-ideals in ordered Γ-semigroups is defined. We prove that intersection of $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-Γ-ideals of $S$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-Γ-ideal but the statement does not hold for union, and in this aim an example is provided. Moreover, we present correspondence between bi-Γ-ideals and $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-Γ-ideals of ordered Γ-semigroups based on level subset and $(\in \vee(\kappa^*, q_\kappa))$-level subset of fuzzy sets.

**Keywords:** ordered Γ-semigroups, fuzzy subsets, $(\in, \in \vee(\kappa^*, q_k))$-fuzzy bi-Γ-ideals

## 1. Introduction

In 1986, Sen and Saha (1986) introduced the notion of a Γ-semigroup. Later on in 1993, the notion of ordered Γ-semigroups was introduced by Sen and Seth (1993). Many classical notions such as ideals, bi-ideals and quasi-ideals in ordered Γ-semigroups and regular ordered Γ-semigroups have been generalized to ordered Γ-semigroups, and these classical notions of ordered Γ-semigroups have been studied by Changphas and Thongkam (2011), Hila (2010), Hila and Pisha (2006), Iampan (2009), Iampan (2015) Kwon and Lee (1998), Mahboob and Khan (2021), and Mahboob et al. (2021).

Zadeh (1965), in 1965, introduced the concept of a fuzzy set. The concept of a fuzzy subgroup introduced by Rosenfeld (1971). In 1979, Kuroki (1979) introduced fuzzy sets in semigroup theory. Fuzzy sets in ordered semigroups were first studied by Kehayopulu and Tsingelis (2002). In Tang (2012), Tang characterized ordered semigroups by $(\in, \in \vee q)$-fuzzy ideals. Later on, the concept of $(\in, \in \vee q_k)$- fuzzy subalgebras in BCK/BCI-algebras is introduced by Jun (2009). In Shabir et al. (2010), characterized the regular semigroups by $(\in, \in \vee q_k)$-fuzzy ideals. In Gambo et al. (2017a), characterized left regular, right regular, regular and completely regular ordered Γ-semigroups in terms of $(\in, \in \vee q_k)$-fuzzy left Γ-ideals, $(\in, \in \vee q_k)$-fuzzy right Γ-ideals and $(\in, \in \vee q_k)$-fuzzy ideals. By generalizing the concept of fuzzy generalized bi Γ-ideals, the concept of $(\in, \in \vee q_k)$-fuzzy bi Γ-ideals in ordered Γ-semigroups is introduced by Gambo et al. (2017b). For More concepts related to this work, we refer readers to Jun et al. (2016); Jun et al. (2014).

In the present work, the concept of $(\in, \in \vee(\kappa^*, q_k))$-fuzzy bi-Γ-ideals in ordered Γ-semigroups is introduced. Furthermore, it is proved that intersection of fuzzy bi-Γ-ideals of $S$ is a fuzzy bi-Γ-ideal but the statement does not hold for union, and in this aim an example is provided. Moreover, the correspondence between bi-Γ-ideals and $(\in, \in \vee(\kappa^*, q_k))$-fuzzy bi-Γ-ideals of ordered Γ-semigroups based on level subset and $(\in \vee(\kappa^*, q_\kappa))$-level subset of fuzzy sets is presented.

## 2. Preliminaries

Let $S$ and $\Gamma$ be the nonempty sets. Then the triplet $(S, \Gamma, \leq)$ is called an ordered Γ-semigroup if $S$ is a Γ-semigroup and $(S, \leq)$ is a partially ordered set such that

$$\tau \leq \vartheta \Rightarrow \tau\gamma\Theta \leq \vartheta\gamma\Theta \text{ and } \Theta\gamma\tau \leq \Theta\gamma\vartheta,$$

for all $\tau, \vartheta, \Theta \in S$ and $\gamma \in \Gamma$.

For a subset $\Omega$ of $S$, we denote $(\Omega] = \{t \in S \mid t \leq a \text{ for some } a \in \Omega\}$. For any nonempty subsets $\Omega$ and $\mho$ of $S$, the following properties hold: (1) $\Omega \subseteq (\Omega]$; (2) $((\Omega]] = (\Omega]$; (3) If $\Omega \subseteq \mho$, then $(\Omega] \subseteq (\mho]$; (4) $(\Omega]\Gamma(\mho] \subseteq (\Omega\Gamma\mho]$ and (5) $((\Omega]\Gamma(\mho]] = (\Omega\Gamma\mho]$.

A subset $T$ of $S$ is said to be a Γ-subsemigroup of $S$ if for all $\tau, \vartheta \in T$ and $\gamma \in \Gamma$, $\tau\gamma\vartheta \in T$. A subsemigroup $B$ of $S$ is called bi-Γ-ideal (briefly, B-Γ-I) of $S$ if $B\Gamma S\Gamma B \subseteq B$ and $(B] \subseteq B$.

A mapping $\mathcal{Q}$ from $S$ to real closed interval [0,1] is called the fuzzy set (briefly, FS) of $S$. For any FSs $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of $S$,

*Corresponding author: G. Muhiuddin, Department of Mathematics, University of Tabuk, Saudi Arabia. Email: chishtygm@gmail.com

© The Author(s) 2022. Published by BON VIEW PUBLISHING PTE. LTD. This is an open access article under the CC BY License (https://creativecommons.org/licenses/by/4.0/).

$\mathcal{Q}_1 \cap \mathcal{Q}_2$, $\mathcal{Q}_1 \cup \mathcal{Q}_2$ and $\mathcal{Q}_1 \circ \mathcal{Q}_2$ are defined as follows:

$$(\mathcal{Q}_1 \cap \mathcal{Q}_2)(\tau) = \min\{\mathcal{Q}_1(\tau), \mathcal{Q}_2(\tau)\} = \mathcal{Q}_1(\tau) \wedge \mathcal{Q}_2(\tau),$$
$$(\mathcal{Q}_1 \cup \mathcal{Q}_2)(\tau) = \max\{\mathcal{Q}_1(\tau), \mathcal{Q}_2(\tau)\} = \mathcal{Q}_1(\tau) \vee \mathcal{Q}_2(\tau),$$

and

$$(\mathcal{Q}_1 \circ \mathcal{Q}_2)(\tau) = \begin{cases} \bigvee_{(\vartheta,\Theta)\in A_\tau} \{\mathcal{Q}_1(\vartheta) \wedge \mathcal{Q}_2(\Theta)\} & \text{if } A_\tau \neq \phi \\ 0 & \text{if } A_\tau = \phi, \end{cases}$$

where $A_\tau = \{(\vartheta, \Theta) \in S \times S \mid \tau \leq \vartheta\alpha\Theta$ for some $\alpha \in \Gamma\}$. Define an order relation $\preceq$ on the set of all FSs of $S$ by

$$\mathcal{Q}_1 \preceq \mathcal{Q}_2 \Leftrightarrow \mathcal{Q}_1(\tau) \leq \mathcal{Q}_2(\tau) \text{ for all } \tau \in S.$$

If $\mathcal{Q}_1, \mathcal{Q}_2$ are FSs of $S$ such that $\mathcal{Q}_1 \preceq \mathcal{Q}_2$, then for each FS $\mathcal{Q}_3$ of $S$, $\mathcal{Q}_1 \circ \mathcal{Q}_3 \preceq \mathcal{Q}_2 \circ \mathcal{Q}_3$ and $\mathcal{Q}_3 \circ \mathcal{Q}_1 \preceq \mathcal{Q}_3 \circ \mathcal{Q}_2$.

A FS $\mathcal{Q}$ of $S$ is called a fuzzy Γ-subsemigroup of $S$ if $\mathcal{Q}(\tau\alpha\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta)\}$ for all $\tau, \vartheta \in S$ and $\alpha \in \Gamma$. A fuzzy Γ-subsemigroup $\mathcal{Q}$ of $S$ is called a fuzzy bi-Γ-ideal of $S$ if (1) $\tau \leq \vartheta \Rightarrow \mathcal{Q}(\tau) \geq \mathcal{Q}(\vartheta)$ and (2) $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta)\}$ for all $\tau, \vartheta, \Theta \in S$ and $\alpha, \beta \in \Gamma$.

## 3. $(\in, \in \vee(\kappa^*, q_k))$-Buzzy Bi-Γ-Ideals of Ordered Γ-Semigroups

Let $\tau \in S$ and $\delta \in (0, 1]$. An ordered fuzzy point $\tau_\delta$ is a mapping from $S$ into [0,1] which is defined as follows:

$$\tau_\delta(\vartheta) = \begin{cases} \delta, \text{ if } \vartheta \in (a], \\ 0, \text{ if } \vartheta \notin (a]. \end{cases}$$

For any FS $\mathcal{Q}$ of $S$, we shall also denote $\tau_\delta \subseteq \mathcal{Q}$ by $\tau_\delta \in \mathcal{Q}$ in the sequel. Then $\tau_\delta \in \mathcal{Q} \Leftrightarrow \mathcal{Q}(\tau) \geq \delta$.

An OFP $\tau_\delta$ of $S$ is said to be quasi-coincident with a FS $\mathcal{Q}$ of $S$, written as $\tau_\delta q\mathcal{Q}$, if $\mathcal{Q}(\tau) + \delta > 1$.

**Definition 3.1.** An OFP $\tau_\delta$ of $S$, for any $\kappa^* \in (0, 1]$, is said to be $(\kappa^*, q)$-quasi-coincident with a FS $\mathcal{Q}$ of $S$, written as $\tau_\delta(\kappa^*, q)\mathcal{Q}$, if

$$\mathcal{Q}(\tau) + \delta > \kappa^*.$$

Let $0 \leq k < \kappa^* \leq 1$. For an OFP $\tau_\delta$, we say that
(1) $\tau_\delta(\kappa^*, q_\kappa)\mathcal{Q}$ if $\mathcal{Q}(\tau) + \delta + k > \kappa^*$;
(2) $\tau_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$ if $\tau_\delta \in \mathcal{Q}$ or $\tau_\delta(\kappa^*, q_\kappa)\mathcal{Q}$;
(3) $\tau_\delta\overline{\alpha}f$ if $\tau_\delta\alpha\mathcal{Q}$ does not hold for $\alpha \in \{(\kappa^*, q_k), \in \vee(\kappa^*, q_k)\}$.

**Definition 3.2.** A FS $\mathcal{Q}$ of $S$ is called an $(\in, \in \vee(\kappa^*, q_k))$-fuzzy Γ-subsemigroup of $S$ if $\tau_\delta \in \mathcal{Q}$ and $\vartheta_\varepsilon \in \mathcal{Q}$ imply $(\tau\gamma\vartheta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$ for all $\tau, \vartheta \in S, \gamma \in \Gamma$ and $\delta, \varepsilon \in (0, 1]$.

**Definition 3.3.** A FS $\mathcal{Q}$ of $S$ is called an $(\in, \in \vee(\kappa^*, q_k))$-fuzzy bi-Γ-ideal (briefly, $(\in, \in \vee(\kappa^*, q_k))$-FBΓI) of $S$ if:

(1) $\tau \leq \vartheta$, $\vartheta_\delta \in \mathcal{Q} \Rightarrow \tau_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$;
(2) $\tau_\delta \in \mathcal{Q}$, $\vartheta_\varepsilon \in \mathcal{Q}$ imply $(\tau\gamma\vartheta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$;
(3) $\tau_\delta \in \mathcal{Q}$, $\Theta_\varepsilon \in \mathcal{Q}$ imply $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$.

for all $\delta, \varepsilon \in (0, 1]$, $\alpha, \beta, \gamma \in \Gamma$ and $\tau, \vartheta, \Theta \in S$.

***Example 3.4.*** Consider an ordered Γ-semigroup $S = \{0, w, b, \vartheta\}$, $\Gamma = \{\alpha, \beta\}$ under the following operations as follows:

| $\alpha$ | 0 | $w$ | $\tau$ | $\vartheta$ | $\beta$ | 0 | $w$ | $\tau$ | $\vartheta$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $w$ | 0 | $\tau$ | 0 | $w$ | $w$ | $w$ | $w$ | $w$ | $w$ |
| $\tau$ | 0 | $\tau$ | 0 | $\vartheta$ | $\tau$ | 0 | 0 | 0 | 0 |
| $\vartheta$ | 0 | 0 | 0 | $\tau$ | $\vartheta$ | $w$ | $w$ | $w$ | $\vartheta$ |

$\leq := \{(0, 0), (w, w), (\tau, \tau), (\vartheta, \vartheta), (0, w), (0, \tau), (0, \vartheta)\}$.

Define $\mathcal{Q} : S \to [0, 1]$ as:

$$\mathcal{Q}(a) = \begin{cases} 0.2 & \text{if } a \in \{0, w, \tau\} \\ 0 & \text{if } a = \vartheta. \end{cases}$$

Take $\kappa^* = 0.5$ and $k = 0.1$. It is easy to verify that $\mathcal{Q}$ is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI of $S$.

**Theorem 3.5.** *Let $\Omega \subseteq S$. Then the $FS_{\chi_\Omega}$ defined as*

$$\chi_\Omega(\tau) = \begin{cases} \frac{\kappa^*-\kappa}{2}, & \text{if } \tau \in \Omega; \\ 0, & \text{if } \tau \notin \Omega, \end{cases}$$

*is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI of S $\Leftrightarrow$ Ω is B-Γ-I of S.*

***Proof.*** Suppose that Ω is B-Γ-I of $S$. Let $\tau, \vartheta \in S$ with $\tau \leq \vartheta$ and $\delta \in (0, 1]$ such that $\vartheta_\delta \in \chi_\Omega$. Then $\chi_\Omega(\vartheta) \geq \delta$. As Ω is a B-Γ-I of $S$, $\tau \in \Omega$. Thus $\chi_\Omega(\tau) = \frac{\kappa^*-\kappa}{2}$. If $\delta \leq \frac{\kappa^*-\kappa}{2}$, then $\chi_\Omega(\tau) \geq \delta$, so $\tau_\delta \in \chi_\Omega$. If $\delta > \frac{\kappa^*-\kappa}{2}$, then $\chi_\Omega(\tau) + \delta > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$. Thus $\tau_\delta(\kappa^*, q_k)\chi_\Omega$. Therefore $\tau_\delta \in \vee(\kappa^*, q_k)\chi_\Omega$.

Next, take any $\tau, \vartheta \in S$ and $\delta, \varepsilon \in (0, 1]$ such that $\tau_\delta, \vartheta_\varepsilon \in \chi_\Omega$. Then $\tau, \vartheta \in \Omega$, $\chi_\Omega(\tau) \geq \delta$, $\chi_\Omega(\vartheta) \geq \varepsilon$. As Ω is a B-Γ-I of $S$, we have $\tau\gamma\vartheta \in \Omega$ for each $\gamma \in \Gamma$. Thus $\chi_\Omega(\tau\gamma\vartheta) \geq \frac{\kappa^*-\kappa}{2}$. If $\min\{\delta, \varepsilon\} \leq \frac{\kappa^*-\kappa}{2}$, then $\chi_\Omega(\tau\gamma\vartheta) \geq \delta$. Therefore $(\tau\gamma\vartheta)_\delta \in \chi_\Omega$. Again, if $\delta > \frac{\kappa^*-\kappa}{2}$, then $\chi_\Omega(\tau\gamma\vartheta) + \delta > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$. So $(\tau\gamma\vartheta)_\delta(\kappa^*, q_k)\chi_\Omega$. Therefore $(\tau\gamma\vartheta)_\delta \in \vee(\kappa^*, q_k)\chi_\Omega$.

Finally, suppose that $\tau, \vartheta, \Theta \in S$ and $\delta, \varepsilon \in (0, 1]$ such that $\tau_\delta, \Theta_\varepsilon \in \chi_\Omega$. Then $\tau, \Theta \in \Omega$, $\chi_\Omega(\tau) \geq \delta$, $\chi_\Omega(\Theta) \geq \varepsilon$. As Ω is a B-Γ-I of $S$, we have $\tau\alpha\vartheta\beta\Theta \in \Omega$ for each $\alpha, \beta \in \Gamma$. Thus $\chi_\Omega(\tau\alpha\vartheta\beta\Theta) \geq \frac{\kappa^*-\kappa}{2}$. If $\min\{\delta, \varepsilon\} \leq \frac{\kappa^*-\kappa}{2}$, then $\chi_\Omega(\tau\alpha\vartheta\beta\Theta) \geq \min\{\delta, \varepsilon\}$. Therefore $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \chi_\Omega$. Again, if $\min\{\delta, \varepsilon\} > \frac{\kappa^*-\kappa}{2}$, then $\chi_\Omega(\tau\alpha\vartheta\beta\Theta) + \min\{\delta, \varepsilon\} > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$. So $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}}(\kappa^*, q_k)\chi_\Omega$. Therefore $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_k)\chi_\Omega$.

Conversely, assume that $\chi_\Omega$ is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI of $S$. Let $\tau, y \in S$ be such that $\tau \leq \vartheta$. If $\vartheta \in \Omega$, then $\chi_\Omega(\vartheta) = \frac{\kappa^*-\kappa}{2}$. As $\chi_\Omega$ is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI of $S$ and $\tau \leq \vartheta$, we have $\chi_\Omega(\tau) \geq \min\{\chi_\Omega(\vartheta), \frac{\kappa^*-\kappa}{2}\} = \frac{\kappa^*-\kappa}{2}$. It follows that $\chi_\Omega(\tau) = \frac{\kappa^*-\kappa}{2}$ and so $\tau \in \Omega$. Let $\vartheta \in S$ and $\tau, \Theta \in \Omega$. Then $\chi_\Omega(\tau) = \frac{\kappa^*-\kappa}{2}$, $\chi_\Omega(\Theta) = \frac{\kappa^*-\kappa}{2}$. Now, we have

$$\chi_\Omega(\tau\alpha\vartheta\beta\Theta) \geq \min\left\{\chi_\Omega(\tau), \chi_\Omega(\Theta), \frac{\kappa^* - \kappa}{2}\right\} = \frac{\kappa^* - \kappa}{2}.$$

Thus $\chi_\Omega(\tau\alpha\vartheta\beta\Theta) = \frac{\kappa^*-\kappa}{2}$ and so $\tau\alpha\vartheta\beta\Theta \in \Omega$. Similarly, $\tau\gamma\vartheta \in \Omega$ for all $\tau, \vartheta \in S$ and $\gamma \in \Gamma$. Hence Ω is a B-Γ-I of $S$. □

**Theorem 3.6.** *A FS $\mathcal{Q}$ of S is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI of S $\Leftrightarrow$*

(1) $\tau \leq \vartheta \Rightarrow \mathcal{Q}(\tau) \geq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$;

(2) $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$;
(3) $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$;

*for all* $\tau, \vartheta, \Theta \in S$ *and* $\alpha, \beta, \gamma \in \Gamma$.

***Proof.*** Let $\mathcal{Q}$ be an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI of $S$ and $\tau, \vartheta \in S$. If $\mathcal{Q}(\tau) < \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for some $\tau, \vartheta \in S$. Choose $\delta \in (0,1]$ such that $\mathcal{Q}(\tau) < \delta \leq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$. Then $\vartheta_\delta \in \mathcal{Q}$, but $(\tau)_\delta \overline{\in \vee(\kappa^*, q_k)}f$, a contradiction. Thus $\mathcal{Q}(\tau) \geq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$. If $\mathcal{Q}(\tau\gamma\vartheta) < \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for some $\tau, \vartheta \in S$ and $\gamma \in \Gamma$. Choose $\delta \in (0,1]$ such that $\mathcal{Q}(\tau\gamma\vartheta) < \delta \leq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$. Then $\tau_\delta, \vartheta_\delta \in \mathcal{Q}$ but $(\tau\gamma\vartheta)_\delta \overline{\in \vee(\kappa^*, q_k)}f$, which is a contradiction. Therefore $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$. Again, If $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) < \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$ for some $\tau, \vartheta, \Theta \in S$ and $\alpha, \beta \in \Gamma$. Choose $\varepsilon \in (0,1]$ such that $\mathcal{Q}(\tau\gamma\vartheta) < \varepsilon \leq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$. Then $\tau_\delta, \Theta_\delta \in \mathcal{Q}$ but $(\tau\alpha\vartheta\beta\Theta)_\delta \overline{\in \vee(\kappa^*, q_k)}f$, which is again a contradiction. Hence $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$.

Conversely, assume that $\mathcal{Q}(\tau) \geq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for all $\tau, \vartheta \in S$. Let $\vartheta_\delta \in \mathcal{Q}$ $(\delta \in (0,1])$. Then $\mathcal{Q}(\vartheta) \geq \delta$. So $\mathcal{Q}(\tau) \geq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\} \geq \min\{\delta, \frac{\kappa^*-\kappa}{2}\}$. If $\delta \leq \frac{\kappa^*-\kappa}{2}$, then $\mathcal{Q}(\tau) \geq \delta$ implies $\tau_\delta \in \mathcal{Q}$. If $\delta > \frac{\kappa^*-\kappa}{2}$, then $\mathcal{Q}(\tau) \geq \frac{\kappa^*-\kappa}{2}$. So $\mathcal{Q}(\tau) + \delta > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$, which implies that $\tau_\delta(\kappa^*, q_\kappa)\mathcal{Q}$. Thus $\tau_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$. Let $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for all $\tau, \vartheta \in S$ and $\gamma \in \Gamma$. Let $\tau_\delta, \vartheta_\varepsilon \in \mathcal{Q}$ $(\delta, \varepsilon \in (0,1])$. Then $\mathcal{Q}(\tau) \geq \delta$ and $\mathcal{Q}(\vartheta) \geq \varepsilon$. So $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\} \geq \min\{\delta, , \varepsilon, \frac{\kappa^*-\kappa}{2}\}$. If $\min\{\delta, \varepsilon\} \leq \frac{\kappa^*-\kappa}{2}$, then $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\delta, \varepsilon\}$ implies $(\tau\gamma\vartheta)_{\min\{\delta,\varepsilon\}} \in \mathcal{Q}$. If $\min\{\delta, \varepsilon\} > \frac{\kappa^*-\kappa}{2}$, then $\mathcal{Q}(\tau\gamma\vartheta) \geq \frac{\kappa^*-\kappa}{2}$. So $\mathcal{Q}(\tau\gamma\vartheta) + \min\{\delta, \varepsilon\} > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$, it follows that $(\tau\gamma\vartheta)_{\min\{\delta,\varepsilon\}}(\kappa^*, q_\kappa)\mathcal{Q}$. Thus $(\tau\gamma\vartheta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$. Also, assume that $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$ for all $\tau, \vartheta \in S$ and $\gamma \in \Gamma$. Let $\tau_\delta, \Theta_\varepsilon \in \mathcal{Q}$ $(\delta, \varepsilon \in (0,1])$. Then $\mathcal{Q}(\tau) \geq \delta$ and $\mathcal{Q}(\Theta) \geq \varepsilon$. So $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\} \geq \min\{\delta, \varepsilon, \frac{\kappa^*-\kappa}{2}\}$. If $\min\{\delta, \varepsilon\} \leq \frac{\kappa^*-\kappa}{2}$, then $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\delta, \varepsilon\}$ implies $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \mathcal{Q}$. If $\min\{\delta, \varepsilon\} > \frac{\kappa^*-\kappa}{2}$, then $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \frac{\kappa^*-\kappa}{2}$. So $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) + \min\{\delta, \varepsilon\} > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$, and thus $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}}(\kappa^*, q_\kappa)\mathcal{Q}$. Therefore $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$ □

**Lemma 3.7.** *Let* $\{\mathcal{Q}_i \mid i \in I, \mathcal{Q}_i$ *is a*$(\in, \in \vee(\kappa^*, q_k)) - FB\Gamma I\}$. *Then* $\bigcap_{i \in I} \mathcal{Q}_i$ *is an* $(\in, \in \vee(\kappa^*, q_k))$*-FBΓI of S.*

***Proof.*** Take any $\tau, \vartheta \in S$ such that $\tau \leq \vartheta$ and $\vartheta_\delta \in \bigcap_{i \in I} \mathcal{Q}_i$. Then $\vartheta_\delta \in \mathcal{Q}_i$ for each $i \in I$. As each $\mathcal{Q}_i$ is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI, $\tau_\delta \in \mathcal{Q}_i$ for each $i \in I$. Thus $\tau_\delta \in \bigcap_{i \in I} \mathcal{Q}_i$.

Next, take any $\tau, \vartheta \in S, \gamma \in \Gamma$ and $\delta, \varepsilon \in (0,1]$ such that $\tau_\delta, \vartheta_\varepsilon \in \bigcap_{i \in I} \mathcal{Q}_i$. Then $\tau_\delta, \vartheta_\varepsilon \in \mathcal{Q}_i$ for each $i \in I$. So $\mathcal{Q}_i(\tau) \geq \delta$, $\geq \mathcal{Q}_i(\vartheta) \geq \varepsilon$. Thus, we have $\bigcap_{i \in I} \mathcal{Q}_i(\tau\gamma\vartheta) = \bigwedge_{i \in I} \mathcal{Q}_i(\tau\gamma\vartheta) \geq \bigwedge_{i \in I} \min\{\mathcal{Q}_i(\tau), \mathcal{Q}_i(\vartheta)\} \geq \min\{\delta, \varepsilon\}$. So $(\tau\gamma\vartheta)_{\min\{\delta,\varepsilon\}} \in \bigcap_{i \in I} \mathcal{Q}_i(\tau\gamma\vartheta)$.

Finally, take any $\tau_\delta \in \bigcap_{i \in I} \mathcal{Q}_i$ and $\Theta_\varepsilon \in \bigcap_{i \in I} \mathcal{Q}_i$ for each $\tau, \Theta \in S$ and $\delta, \varepsilon \in (0,1]$. Therefore $\tau_\delta \in \mathcal{Q}_i$ and $\Theta_\varepsilon \in \mathcal{Q}_i$ for $i \in I$. As each $\mathcal{Q}_i$ is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI, so for all $\alpha, \beta \in \Gamma$, $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_\kappa)\mathcal{Q}_i, \forall\ i \in I$. Thus $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \mathcal{Q}_i$ or $\mathcal{Q}_i(\tau\alpha\vartheta\beta\Theta) + \min\{\delta, \varepsilon\} + \kappa \geq \kappa^*$. If $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \mathcal{Q}_i$, then

$$\bigcap_{i \in I} \mathcal{Q}_i(\tau\alpha\vartheta\beta\Theta) = \bigwedge_{i \in I} \mathcal{Q}_i(\tau\alpha\vartheta\beta\Theta) \geq \bigwedge_{i \in I} \min\{\delta, \varepsilon\} = \min\{\delta, \varepsilon\}.$$

Therefore $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \bigcap_{i \in I} \mathcal{Q}_i$, which implies that $(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_\kappa) \bigcap_{i \in I} \mathcal{Q}_i$. Similarly, if $\mathcal{Q}_i(\tau\alpha\vartheta\beta\Theta) + \min\{\delta, \varepsilon\} + \kappa \geq \kappa^*$, then

$$(\tau\alpha\vartheta\beta\Theta)_{\min\{\delta,\varepsilon\}} \in \vee(\kappa^*, q_\kappa) \bigcap_{i \in I} \mathcal{Q}_i.$$

Hence $\bigcap_{i \in I} \mathcal{Q}_i$ is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI. □

**Remark 3.8.** Let $\{\mathcal{Q}_i | i \in I, \mathcal{Q}_i$ *is a*$(\in, \in \vee(\kappa^*, q_k)) - FB\Gamma I\}$. Then $\bigcup_{i \in I} \mathcal{Q}_i$ need not be an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI. The following example validates the above claim:

***Example 3.9.*** Consider an ordered Γ-semigroup $S = \{w, \tau, \vartheta, \Theta\}$, $\Gamma = \{\alpha\}$ under the following operations as follows:

| $\alpha$ | $w$ | $\tau$ | $\vartheta$ | $\Theta$ |
|---|---|---|---|---|
| $w$ | $w$ | $w$ | $w$ | $w$ |
| $\tau$ | $w$ | $w$ | $\Theta$ | $w$ |
| $\vartheta$ | $w$ | $w$ | $w$ | $w$ |
| $\Theta$ | $w$ | $w$ | $w$ | $w$ |

$$\leq := \{(w, w), (\tau, \tau), (\vartheta, \vartheta), (\Theta, \Theta), (w, \tau)\}.$$

Define $\mathcal{Q}_1$ and $\mathcal{Q}_2$ as follows:

$$\mathcal{Q}_1(w) = 0.4, \mathcal{Q}_1(\tau) = 0.4, \mathcal{Q}_1(\vartheta) = 0, \mathcal{Q}_1(\Theta) = 0;$$
$$\mathcal{Q}_2(w) = 0.4, \mathcal{Q}_2(\tau) = 0, \mathcal{Q}_2(\vartheta) = 0.4, \mathcal{Q}_2(\Theta) = 0.$$

Then $\mathcal{Q}_1, \mathcal{Q}_2$ are $(\in, \in \vee(\kappa^*, q_k))$-FBΓIs of $S$, but $\mathcal{Q}_1 \cup \mathcal{Q}_2$ is not a $(\in, \in \vee(\kappa^*, q_k))$-FBΓI because $0 = \mathcal{Q}_1(\Theta) \vee \mathcal{Q}_2(\Theta) = (\mathcal{Q}_1 \cup \mathcal{Q}_2)(\Theta) = (\mathcal{Q}_1 \cup \mathcal{Q}_2)(\tau\alpha\vartheta) < min\{(\mathcal{Q}_1 \cup \mathcal{Q}_2)(\tau), (\mathcal{Q}_1 \cup \mathcal{Q}_2)(\vartheta), \frac{\kappa^*-\kappa}{2}\}$.

**Definition 3.10.** Let $\mathcal{Q}$ be any FS of S. For any $\delta \in (0,1]$, the set

$$U(\mathcal{Q}; \delta) = \{\tau \in S \mid \mathcal{Q}(\tau) \geq \delta\},$$

is called a level subset of $\mathcal{Q}$.

**Theorem 3.11.** *Let* $\mathcal{Q}$ *be a FS of S. Then* $\mathcal{Q}$ *is an* $(\in, \in \vee(\kappa^*, q_k))$*-FBΓI of S* $\Leftrightarrow$ $U(\mathcal{Q}; \delta)(\neq \emptyset) \sim (\delta \in (0, \frac{\kappa^*-\kappa}{2}])$ *is a B-Γ-I of S.*

***Proof.*** Suppose that $\mathcal{Q}$ is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI of $S$. Let $\tau, \vartheta \in S$ be such that $\tau \leq \vartheta \in U(\mathcal{Q}; \delta)$, where $\delta \in (0, \frac{\kappa^*-\kappa}{2}]$. Then $\mathcal{Q}(\vartheta) \geq \delta$. By Theorem 3.6, $\mathcal{Q}(\tau) \geq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\} \geq \min\{\delta, \frac{\kappa^*-\kappa}{2}\} = \delta$. Therefore $\tau \in U(\mathcal{Q}; \delta)$. Let $\tau, \vartheta \in U(\mathcal{Q}; \delta)$. Then $\mathcal{Q}(\tau) \geq \delta$ and $\mathcal{Q}(y) \geq \delta$. So, by Theorem 3.6, $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\} \geq \min\{\delta, \delta, \frac{\kappa^*-\kappa}{2}\} = \delta$. Thus $\mathcal{Q}(\tau\gamma\vartheta) \geq \delta$. Therefore $\tau\gamma\vartheta \in U(\mathcal{Q}; \delta)$. Assume that $\vartheta \in S$ and $\tau, \Theta \in U(\mathcal{Q}; \delta)$. Then $\mathcal{Q}(\tau) \geq \delta$ and $\mathcal{Q}(\Theta) \geq \delta$. So, by Theorem 3.6, $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\} \geq \min\{\delta, \delta, \frac{\kappa^*-\kappa}{2}\} = \delta$. Thus $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \delta$. Therefore $\tau\alpha\vartheta\beta\Theta \in U(\mathcal{Q}; \delta)$. Hence $U(\mathcal{Q}; \delta)$ is a B-Γ-I.

Conversely, assume that $U(\mathcal{Q};\delta)(\neq \emptyset)$ is a B-Γ-I of $S$ for all $\delta \in (0, \frac{\kappa^*-\kappa}{2}]$. Take any $\tau, \vartheta \in S$ with $\tau \leq \vartheta$. If $\mathcal{Q}(\tau) < \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$. Then $\mathcal{Q}(\tau) < \delta \leq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$, for some $\delta \in (0, \frac{\kappa^*-\kappa}{2}]$. It follows that $\vartheta \in U(\mathcal{Q};\delta)$ but $\tau \notin U(\mathcal{Q};\delta)$, which is not possible. Thus $\mathcal{Q}(\tau) \geq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for all $\tau, \vartheta \in S$ with $\tau \leq \vartheta$. If $\mathcal{Q}(\tau\gamma\vartheta) < \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta) \frac{\kappa^*-\kappa}{2}\}$ for some $\tau, \vartheta \in S$. Therefore there exists $\delta \in (0, \frac{\kappa^*-\kappa}{2}]$ such that $\mathcal{Q}(\tau\gamma\vartheta) < \delta \leq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ implies $\tau_\delta, \vartheta_\delta \in U(\mathcal{Q};\delta)$ but $(\tau\gamma\vartheta)_\delta \notin U(\mathcal{Q};\delta)$, which is a contradiction. Thus $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for all $\tau, \vartheta \in S$. Again, if $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) < \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta) \frac{\kappa^*-\kappa}{2}\}$ for some $\tau, \vartheta, \Theta \in S$ and $\alpha, \beta \in \Gamma$. Therefore there exist $\delta \in (0, \frac{\kappa^*-\kappa}{2}]$ such that $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) < \delta \leq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$ implies $\tau_\delta, \Theta_\delta \in U(\mathcal{Q};\delta)$ but $(\tau\alpha\vartheta\beta\Theta)_\delta \notin U(\mathcal{Q};\delta)$, which is again a contradiction, and hence, $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$ for all $\tau, \vartheta, \Theta \in S$. Hence by Theorem 3.6, $\mathcal{Q}$ is an $(\in, \in \vee(\kappa^*, q_k))$-FBΓI of $S$.

**Definition 3.12.** Let $\mathcal{Q}$ be a FS of S. The set $[\mathcal{Q}]_\delta = \{\tau \in S \mid \tau_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}\}$, where $\delta \in (0,1]$, is called an $(\in \vee(\kappa^*, q_\kappa))$-level subset of $\mathcal{Q}$.

**Theorem 3.13.** *Let $\mathcal{Q}$ be a FS of $S$ such that $\tau \leq \vartheta$ implies $\mathcal{Q}(\tau) \geq \mathcal{Q}(\vartheta)$. Then $\mathcal{Q}$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-FBΓI of $R \Leftrightarrow \forall\, \delta \in (0,1]$, the $(\in \vee(\kappa^*, q_\kappa))$-level subset $[\mathcal{Q}]_\delta$ of $\mathcal{Q}$ is a B-Γ-I of R.*

***Proof.*** $(\Rightarrow)$ Take any $\tau \in S$ and $\vartheta \in [\mathcal{Q}]_\delta$ such that $\tau \leq \vartheta$. As $\vartheta \in [\mathcal{Q}]_\delta$, we have $\vartheta_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$ implies $\mathcal{Q}(\vartheta) \geq \delta$ or $\mathcal{Q}(\vartheta) + \delta + \kappa > \kappa^*$. By hypothesis, we have $\mathcal{Q}(\tau) \geq \mathcal{Q}(\vartheta) \geq \delta$ or $\mathcal{Q}(\tau) \geq \mathcal{Q}(\vartheta) \geq \kappa^* - \delta - \kappa$. Thus $\tau_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$. Therefore $\tau \in [\mathcal{Q}]_\delta$. Next, take any $\tau, \vartheta \in [\mathcal{Q}]_\delta$. Then $\tau_\delta, \vartheta_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$; that is $\mathcal{Q}(\tau) \geq \delta$ or $\mathcal{Q}(\tau) + \delta + \kappa > \kappa^*$ and $\mathcal{Q}(\vartheta) \geq \delta$ or $\mathcal{Q}(\vartheta) + \delta + \kappa > \kappa^*$.

**Case (i).** Let $\mathcal{Q}(\tau) \geq \delta$ and $\mathcal{Q}(\vartheta) \geq \delta$. If $\delta > \frac{\kappa^*-\kappa}{2}$, then

$$\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\} \geq \min\{\delta, \delta, \frac{\kappa^*-\kappa}{2}\} = \frac{\kappa^*-\kappa}{2},$$

and, thus, $(\tau\gamma\vartheta)_\delta(\kappa^*, q_\kappa)\mathcal{Q}$. If $\delta \leq \frac{\kappa^*-\kappa}{2}$, then

$$\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\} \geq \min\{\delta, \delta, \frac{\kappa^*-\kappa}{2}\} = \delta,$$

and so $(\tau\gamma\vartheta)_\delta \in \mathcal{Q}$. Hence $(\tau\gamma\vartheta)_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$.

**Case (ii).** Let $\mathcal{Q}(\tau) \geq \delta$ and $\mathcal{Q}(\vartheta) + \delta + \kappa > \kappa^*$. If $\delta > \frac{\kappa^*-\kappa}{2}$, then

$$\begin{aligned}\mathcal{Q}(\tau\gamma\vartheta) &\geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}\\ &= \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}\\ &> \min\{(\kappa^*-\delta-\kappa), \frac{\kappa^*-\kappa}{2}\}\\ &= \kappa^*-\delta-\kappa,\end{aligned}$$

that is $\mathcal{Q}(\tau\gamma\vartheta) + \delta + \kappa > \kappa^*$, and thus $(\tau\gamma\vartheta)_\delta(\kappa^*, q_\kappa)\mathcal{Q}$. If $\delta \leq \frac{\kappa^*-\kappa}{2}$, then

$$\begin{aligned}\mathcal{Q}(\tau\gamma\vartheta) &\geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}\\ &\geq \min\{\delta, (\kappa^*-\delta-\kappa), \frac{\kappa^*-\kappa}{2}\} = \delta,\end{aligned}$$

and so $(\tau\gamma\vartheta)_\delta \in \mathcal{Q}$. Hence $(\tau\gamma\vartheta)_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$.

**Case (iii).** Let $\mathcal{Q}(\tau) + \delta + \kappa > \kappa^*$ and $\mathcal{Q}(\vartheta) \geq \delta$. Proof is similar to the proof of Case (ii).

**Case (iv).** Let $\mathcal{Q}(\tau) + \delta + \kappa > \kappa^*$ and $\mathcal{Q}(\vartheta) + \delta + \kappa > \kappa^*$. Proof is similar to previous two cases.

Thus in each case, we have $(\tau\gamma\vartheta)_\delta \in \vee(\kappa^*, q_\kappa)\mathcal{Q}$, and so $\tau\gamma\vartheta \in [\mathcal{Q}]_\delta$. Similarly, for any $\vartheta \in R$ and $\tau, \Theta \in [\mathcal{Q}]_\delta$, we have $\tau\alpha\vartheta\beta\Theta \in [\mathcal{Q}]_\delta$. Hence $[\mathcal{Q}]_\delta$ is a B-Γ-I of $R$.

$(\Leftarrow)$ Let $\mathcal{Q}(\tau) < \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for some $\tau, \vartheta \in R$. Then $\delta \in (0, \frac{\kappa^*-\kappa}{2}]$ such that $\mathcal{Q}(\tau) < \delta \leq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$. It follows that $\vartheta \in [\mathcal{Q}]_\delta$ but $\tau \notin [\mathcal{Q}]_\delta$ which is a condradictioin, and hence $\mathcal{Q}(\tau) \geq \min\{\mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$. Let $\mathcal{Q}(\tau\gamma\vartheta) < \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for some $\tau, \vartheta \in R$. Then $\exists$ $\delta \in (0, \frac{\kappa^*-\kappa}{2}]$ such that $\mathcal{Q}(\tau\gamma\vartheta) < \delta \leq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$. It follows that $\tau, \vartheta \in [\mathcal{Q}]_\delta$ but $\tau\gamma\vartheta \notin [\mathcal{Q}]_\delta$ which is a condradictioin. Therefore $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for all $\tau, \vartheta \in R$. Similarly, $\mathcal{Q}(\tau\gamma\vartheta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\vartheta), \frac{\kappa^*-\kappa}{2}\}$ for all $\tau, \vartheta \in R$. Next, suppose that $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$ for some $\tau, \vartheta, \Theta \in R$. Then $\tau, \Theta \in [\mathcal{Q}]_\delta$ but $\tau\alpha\vartheta\beta\Theta \notin [\mathcal{Q}]_\delta$ which is again a contraduction. Thus $\mathcal{Q}(\tau\alpha\vartheta\beta\Theta) \geq \min\{\mathcal{Q}(\tau), \mathcal{Q}(\Theta), \frac{\kappa^*-\kappa}{2}\}$. Hence $\mathcal{Q}$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-FBΓI of $S$.

## 4. Conclusion

The main purpose of the present paper is to introduce the concept of $(\in, \in \vee(k^*, q_k))$-fuzzy bi-Γ-ideals in ordered Γ-semigroups by generalizing the concept of $(\in, \in \vee q_k)$-fuzzy bi-Γ-ideals. Equivalent condition investigated for $(\in, \in \vee(k^*, q_k))$-fuzzy bi-Γ-ideals in ordered Γ-semigroups. Furthermore, we have proven that intersection of $(\in, \in \vee(k^*, q_k))$-fuzzy bi-Γ-ideals of $S$ is an $(\in, \in \vee(k^*, q_k))$-fuzzy bi-Γ-ideal but the statement is not valid for union, and in this aim an example is provided. Moreover, we presented correspondence between bi-Γ-ideals and $(\in, \in \vee(\kappa^*, q_k))$-fuzzy bi-Γ-ideals of ordered Γ-semigroups based on level subset and $(\in \vee(\kappa^*, q_\kappa))$-level subset of fuzzy sets. In our future work, by using the concept of $(k^*, q)$-quasi-coincident with a fuzzy subset of ordered Γ-semigroups, some different kinds of ideals will be introduced.


## Acknowledgements

The authors are grateful to the anonymous referees for a careful checking of the details and for helpful comments that improved this paper.


## Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

## References

Changphas, T., & Thongkam, B. (2011). A note on maximal ideals in ordered Γ-semigroups. *International Mathematical Forum*, *6*(67), 3343–3347.

Gambo, I., Sarmin, N. H., Khan, H. U., & Khan, F.M. (2017a). The characterization of regular ordered Γ-semigroups in terms of $(\in, \in \vee q_k)$-fuzzy Γ-ideals. *Malaysian Journal of Fundamental and Applied Sciences*, *13*(4), 576–580. https://doi.org/10.11113/mjfas.v0n0.608

Gambo, I., Sarmin, N. H., Khan, H. U., & Khan, F. M. (2017b). New fuzzy generalized bi Γ-ideals of the type $(\in, \in \vee q_k)$ in ordered Γ-semigroups. *Malaysian Journal of Fundamental and Applied Sciences*, *13*(4), 666–670. https://doi.org/10.11113/mjfas.v13n4.756

Hila, K. (2010). On quasi-prime, weakly quasi-prime left ideals in ordered-Γ-semigroups. *Mathematica Slovaca*, *60*(2), 195–212. https://doi.org/10.2478/s12175-010-0006-x

Hila, K., & Pisha, E. (2006). Characterizations on ordered Γ-semigroup. *International Journal of Pure and Applied Mathematics*, *28*, 423–439.

Iampan, A. (2009). Characterizing ordered bi-ideals in ordered Γ-semigroups. *Iranian Journal of Mathematical Sciences and Informatics*, *4*, 17–25.

Iampan, A. (2015). Characterizing intuitionistic fuzzy Γ-Ideals of ordered Γ-semigroups by means of intuitionistic fuzzy points. *Notes on Intuitionistic Fuzzy Sets*, *21*(3), 24–39.

Jun, Y. B. (2009). Generalizations of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras. *Computers and Mathematics with Applications*, *58*(7), 1383–1390. https://doi.org/10.1016/j.camwa.2009.07.043

Jun, Y. B., Song, S. Z., & Muhiuddin, G. (2014). Concave soft sets, critical soft points, and union-soft ideals of ordered semigroups. *The Scientific World Journal, 2014*. https://doi.org/10.1155/2014/467968

Jun, Y. B., Song, S. Z., & Muhiuddin, G. (2016). Hesitant fuzzy semigroups with a frontier. *Journal of Intelligent and Fuzzy Systems*, *30*(3), 1613–1618. https://doi.org/10.3233/ifs-151869

Kehayopulu, N., & Tsingelis, M. (2002). Fuzzy sets in ordered groupoids. *Semigroup Forum*, *65*, 128–132.

Kwon, Y. I., & Lee, S. K. (1998). The weakly prime ideals of ordered Γ-semigroups. *Communications of the Korean Mathematical Society*, *13*, 251–256.

Kuroki, N. (1979). Fuzzy bi-ideals in semigroups. *Commentarii mathematici Universitatis Sancti Pauli*, *28*, 17–21.

Mahboob, A., Davvaz, B., & Khan, N. M. (2021). Ordered Γ-semigroups and fuzzy Γ-ideals. *Iranian Journal of Mathematical Sciences and Informatics*, *16*(2), 145–162. http://ijmsi.ir/article-1-1295-en.html

Mahboob, A., & Khan, N. M. (2021). Pure Γ-ideals in Γ-semigroups. *Afrika Matematika*, *32*(7-8), 1201–1210. https://doi.org/10.1007/s13370-021-00893-7

Rosenfeld, A. (1971). Fuzzy subgroups. *Journal of Mathematical Analysis and Applications*, *35*, 512–517. https://doi.org/10.1016/0022-247X(71)90199-5

Shabir, M., Jun, Y. B., & Nawaz, Y. (2010). Characterizations of regular semigroups by $(\alpha, \beta)$-fuzzy ideals. *Computers and Mathematics with Applications*, *59*(1), 161–175. https://doi.org/10.1016/j.camwa.2009.07.062

Sen, M. K., & Saha, N. K. (1986). On Γ-semigroup I. *Bulletin of the Calcutta Mathematical Society*, *78*, 181–186.

Sen, M. K., & Seth, A. (1993). On po-Γ-semigroups. *Bulletin of the Calcutta Mathematical Society*, *85*, 445–450.

Tang, J. (2012). Characterization of ordered Γ-semigroups by of $(\in, \in \vee q)$-fuzzy ideals. *The World Academy of Science, Engineering and Technology*, *6*, 518–530.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*, 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

**How to Cite:** Muhiuddin, G., Mahboob, A., & Elnair, M. E.A. (2022). A New Study Based on Fuzzy Bi-Γ-Ideals in Ordered-Γ-Semigroups. *Journal of Computational and Cognitive Engineering* 1(1), 42–46, https://doi.org/10.47852/bonviewJCCE19919205514