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Refined Pythagorean Fuzzy Sets: Properties, Set-Theoretic Operations and Axiomatic Results

Muhammad Saeed[1], **Muhammad Rayees Ahmad**[1] **and Atiqe Ur Rahman**[1,*]

[1]*Department of Mathematics, University of Management and Technology, Pakistan*

Abstract: Refined Pythagorean fuzzy set is the generalization of fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set, refined fuzzy set, and refined intuitionistic fuzzy set as it not only bears the characteristics of these models but also addresses their limitations for real-life scenarios with sub-grades corresponding to membership and non-memberships grades. In order to equip the existing literature with conceptual framework of refined Pythagorean fuzzy sets, essential properties, set-theoretic operations, and axiomatic results are characterized with the help of numerical examples.

Keywords: fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set

1. Introduction

Fuzzy set (*f*-set) (Zadeh, 1965) and intuitionistic fuzzy set (*if*-set) (Atanassov, 1986) are initiated to tackle uncertain data and information. In *f*-set, the condition "well defined" of classical set is characterized by a membership function μ_T defined by a membership grade $\mu_T(u_i)$ within [0,1] for all members u_i of initial universe $\mathcal{U}$, whereas *if*-set characterizes such condition by two functions, that is, membership function μ_T and non-membership function μ_F defined by membership grade $\mu_T(u_i)$ and non-membership grade $\mu_F(u_i)$, respectively, within [0,1] for all $u_i \in \mathcal{U}$ subject to conditions: (i) both $\mu_T(u_i)$ and $\mu_F(u_i)$ are dependent and (ii) their sum $\mu_T(u_i) + \mu_F(u_i)$ must lie within [0,1] with hesitancy grade $\mu_H(u_i) = 1 - (\mu_T(u_i) + \mu_F(u_i))$. The concept of *if*-set is not capable to tackle the situation when $\mu_T(u_i) = 0.6$ and $\mu_F(u_i) = 0.7$; therefore, Yager (2013) and Yager & Abbasov (2013) initiated the concept of Pythagorean fuzzy set (*pf*-set) which considers both μ_T and μ_F with conditions of $0 \leq \mu_T^2(u_i) + \mu_F^2(u_i) \leq 1$ and $\mu_H(u_i) = 1 - \sqrt{\mu_T^2(u_i) + \mu_F^2(u_i)}$. This model is more flexible and reliable as compared to *if*-set to deal with uncertain nature of data. Zhang & Xu (2014) discussed Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) technique for multi-criteria decision-making (MCDM) based on *pf*-sets. Peng & Yang (2015) and Yager (2016) investigated the elementary properties and operations of *pf*-sets. Garg (2016) determined correlation coefficients for *pf*-sets and discussed its properties. Xiao & Ding (2019), Wei & Wei (2018), Li & Lu (2019), and Zhang (2019) calculated various similarity and distance measures between *pf*-sets. Thao & Smarandache (2019) measured the entropy of *pf*-sets. Many researchers investigated structural contributions for uncertain environment, but the contributions of authors (Ihsan et al., 2021; Kamac et al., 2021; Petchimuthu et al., 2020; Rahman et al., 2021) are more significant. The researchers (Gao & Deng, 2021; Ullah et al., 2020; Zhou et al., 2020) have discussed the applications of *pf*-sets in different fields of mathematical sciences. Recently, the uncertain complexities in different daily-life problems have been investigated and addressed through employing various MCDM techniques by the authors (Bakioglu & Atahan, 2021; Chen, 2018; Chen, 2020; Farhadinia, 2021; Ejegwa, 2020; Mahmood & Ali, 2021; Rani et al., 2019; Xie et al., 2021). In many real-life scenarios, we encounter with some situations which demand sub-grades corresponding to membership and non-membership degrees; therefore, the concepts of *f*-set, *if*-set, and *pf*-set have further extended to their respective refined versions by Smarandache (2019). Rahman et al. (2020) characterized the rudiments of refined intuitionistic fuzzy sets by employing abstract and analytical approaches. Dealing with uncertain data is always a challenging field of study for the researchers. Some models have already been devised to tackle the scenarios with uncertainties and vagueness. Pythagorean fuzzy set is considered notable in this regards. In real-world perspective scenarios, there are various situations which demand multi-grades corresponding to membership and non-membership grades which are addressed by existing models; therefore, Smarandache initiated the concept of refined Pythagorean fuzzy set. The basic properties and set-theoretic operations of *rpf*-set are characterized in this article to provide a framework for further utilization in different fields of study.

*Corresponding author: Atiqe Ur Rahman, Department of Mathematics, University of Management and Technology, Pakistan. Email: aurkhb@gmail.com

2. Preliminaries

In this part of the paper, certain definitions of elementary nature are reviewed from existing literature to support the main results. Throughout the paper, $\breve{\mathfrak{U}}$, $\mathcal{I}$, T_η^ω, and F_η^λ represent initial universe, unit closed interval, sub-membership grades, and sub-non-membership grades, respectively.

Definition 2.1. Fuzzy Set(*f*-set) (Zadeh, 1965) The *f*-set $\check{\zeta}_f = \{< \hat{u}, \alpha_{\check{\zeta}}(\hat{u}) > | \hat{u} \in \breve{\mathfrak{U}}\}$ on $\breve{\mathfrak{U}}$ such that

$$\alpha_{\check{\zeta}} : \breve{\mathfrak{U}} \rightarrow \check{\mathcal{I}}$$

where $\alpha_{\check{\zeta}}(\hat{u})$ denotes the belonging degree of $\hat{u} \in \breve{\mathfrak{U}}$.

Example 2.2. Katherine wants to purchase a laptop for her educational research purpose. She has to evaluate a unique laptop which has all the specifications of a standard laptop. Let $\breve{\mathfrak{U}} = \{\check{\mathcal{L}}_1, \check{\mathcal{L}}_2, \check{\mathcal{L}}_3, \check{\mathcal{L}}_4\}$ be different brands of laptops such that

$$\begin{aligned} \check{\mathcal{L}}_1 &= \text{Apple}, \\ \check{\mathcal{L}}_2 &= \text{Samsung}, \\ \check{\mathcal{L}}_3 &= \text{Hewlett} - \text{Packard (HP)}, \\ \check{\mathcal{L}}_4 &= \text{Lenovo}. \end{aligned}$$

Then *f*-set $\check{\zeta}_f$ over $\breve{\mathfrak{U}}$ can be stated as

$$\check{\zeta}_f = \{< \check{\mathcal{L}}_1, 0.45 >, < \check{\mathcal{L}}_2, 0.57 >, < \check{\mathcal{L}}_3, 0.6 >, < \check{\mathcal{L}}_4, 0.64 >\}$$

Definition 2.3. Intuitionistic Fuzzy Set(*if*-set) (Atanassov, 1986) An *if*-set $\check{\zeta}_{IF}$ over $\breve{\mathfrak{U}}$ is stated as

$$\check{\zeta}_{IF} = \{< \hat{u}, T_{\check{\zeta}}(\hat{u}), F_{\check{\zeta}}(\hat{u}) > | \quad \hat{u} \in \breve{\mathfrak{U}}\},$$

with $T_{\check{\zeta}}(\hat{u}), F_{\check{\zeta}}(\hat{u}) : \breve{\mathfrak{U}} \rightarrow \check{\mathcal{I}}$, and $0 \leq T_{\check{\zeta}}(\hat{u}) + F_{\check{\zeta}}(\hat{u}) \leq 1$. The hesitant grade is

$$I_{\check{\zeta}}(\hat{u}) = 1 - T_{\check{\zeta}}(\hat{u}) - F_{\check{\zeta}}(\hat{u}).$$

Example 2.4. Consider the Example 2.2, the *if*-set $\check{\zeta}_{IF}$ over $\breve{\mathfrak{U}}$ is presented as

$$\begin{aligned} \check{\zeta}_f = \{&< \check{\mathcal{L}}_1, 0.75, 0.14 >, < \check{\mathcal{L}}_2, 0.57, 0.2 >, < \check{\mathcal{L}}_3, 0.6, 0.3 >, \\ &< \check{\mathcal{L}}_4, 0.64, 0.16 >\} \end{aligned}$$

Definition 2.5. Pythagorean Fuzzy Set(*pf*-set) (Yager, 2013; Yager & Abbasov, 2013)
A *pf*-set $\check{\zeta}_{PF}$ over $\breve{\mathfrak{U}}$ is characterized as

$$\check{\zeta}_{PF} = \{< \hat{u}, T_{\check{\zeta}}(\hat{u}), F_{\check{\zeta}}(\hat{u}) > | \hat{u} \in \breve{\mathfrak{U}}\},$$

with $T_{\check{\zeta}}, F_{\check{\zeta}} : \breve{\mathfrak{U}} \rightarrow \check{\mathcal{I}}$ and $0 \leq (T_{\check{\zeta}}(\hat{u}))^2 + (F_{\check{\zeta}}(\hat{u}))^2 \leq 1$. The degree of hesitancy is

$$I_{\check{\zeta}}(\hat{u}) = \sqrt{1 - (T_{\check{\zeta}}(\hat{u}))^2 - (F_{\check{\zeta}}(\hat{u}))^2}$$

Example 2.6. Consider the Example 2.2, the *pf*-set $\check{\zeta}_{PF}$ over $\breve{\mathfrak{U}}$ is stated as

$$\begin{aligned} \check{\zeta}_f = \{&< \check{\mathcal{L}}_1, 0.5625, 0.0196 >, < \check{\mathcal{L}}_2, 0.3249, 0.04 >, \\ &< \check{\mathcal{L}}_3, 0.36, 0.09 >, < \check{\mathcal{L}}_4, 0.4096, 0.0256 >\} \end{aligned}$$

Definition 2.7. Refined Pythagorean Fuzzy Set(*rpf*-set) (Smarandache, 2019)
A refined Pythagorean fuzzy set(*rpf*-set) $\check{\zeta}_{prfs}$ in $\breve{\mathfrak{U}}$ is given by

$$\check{\zeta}_{prfs} = \{< \hat{u}, T^{\omega}_{\check{\zeta}}(\hat{u}), F^{\lambda}_{\check{\zeta}}(\hat{u}) >: \quad \omega \in N^{\alpha}_1, \quad \lambda \in N^{\beta}_1, \quad \alpha + \beta \geq 3, \quad \hat{u} \in \breve{\mathfrak{U}}\},$$

where $\alpha, \beta \in N$ such that $T^{\omega}_{\check{\zeta}}, F^{\lambda}_{\check{\zeta}} : \breve{\mathfrak{U}} \rightarrow \check{\mathcal{I}}$, with the condition that, $0 \leq \sum_{\omega=1}^{\alpha} (T^{\omega}_{\check{\zeta}})^2 + \sum_{\lambda=1}^{\beta} (F^{\lambda}_{\check{\zeta}})^2 \leq 1$, such that the refined hesitancy degree is

$$I_{\check{\zeta}}(\hat{u}) = \sqrt{1 - \sum_{\omega=1}^{\alpha} (T^{\omega}_{\check{\zeta}})^2 - \sum_{\lambda=1}^{\beta} (F^{\lambda}_{\check{\zeta}})^2} \quad \in [0, 1].$$

It is denoted by $(\hat{u}, \check{\mathcal{Q}})$, where $\check{\mathcal{Q}} = (T_{\check{\zeta}}, F_{\check{\zeta}})$.
Note: Here "N" denotes the set of natural number.

Example 2.8. Consider the Example 2.2, the *rpf*-set $\check{\zeta}_{prfs}$ can be written in such a way that

$$\begin{aligned} \check{\zeta}_{prfs} = \{ &< \check{\mathcal{L}}_1, (0.02, 0.1), (0.3, 0.5) >, < \check{\mathcal{L}}_2, (0.05, 0.34), (0.45, 0.04) >, \\ &< \check{\mathcal{L}}_3, (0.01, 0.6), (0.3, 0.02) >, < \check{\mathcal{L}}_4, (0.6, 0.04), (0.12, 0.2) > \}. \end{aligned}$$

3. Basic Notions of *rpf*-set

In this segment, the elementary essential properties of *rpf*-set are investigated with examples.

Definition 3.1. Refined Pythagorean fuzzy subset. Let $\breve{\eta}^1{}_{rpfs} = (\hat{u}, \check{\mathcal{Q}}_1)$ and $\breve{\eta}^2{}_{rpfs} = (\hat{u}, \check{\mathcal{Q}}_2)$ be two *rpf*-sets, then $\breve{\eta}^1{}_{rpfs} \subseteq \breve{\eta}^2{}_{rpfs}$ if $\sum_{\omega=1}^{\alpha} T^{\omega}_{\breve{\eta}_1}(\hat{u}) \leq \sum_{\omega=1}^{\alpha} T^{\omega}_{\breve{\eta}_2}(\hat{u}), \quad \sum_{\lambda=1}^{\beta} F^{\lambda}_{\breve{\eta}_1}(\hat{u}) \geq \sum_{\lambda=1}^{\beta} F^{\lambda}_{\breve{\eta}_2}(\hat{u}), \ \forall \quad \hat{u} \in \breve{\mathfrak{U}}$.

Remark 3.2. 1. If $\sum_{\omega=1}^{\alpha} T^{\omega}_{\breve{\eta}_1}(\hat{u}) < \sum_{\omega=1}^{\alpha} T^{\omega}_{\breve{\eta}_2}(\hat{u}), \quad \sum_{\lambda=1}^{\beta} F^{\lambda}_{\breve{\eta}_1}(\hat{u}) > \sum_{\lambda=1}^{\beta} F^{\lambda}_{\breve{\eta}_2}(\hat{u}) \quad \forall \quad \hat{u} \in \breve{\mathfrak{U}}$. It is denoted by $(\hat{u}, \check{\mathcal{Q}}_1) \subset (\hat{u}, \check{\mathcal{Q}}_2)$.

2. Suppose $(\hat{u}, \check{\mathcal{Q}}_1{}^i)$ and $(\hat{u}, \check{\mathcal{Q}}_2{}^i)$ be two family of *rpf*-sets, then $(\hat{u}, \check{\mathcal{Q}}_1{}^i)$ is called family of Pythagorean fuzzy subset of $(\hat{u}, \check{\mathcal{Q}}_2{}^i)$ if $\check{\mathcal{Q}}_1{}^i \subset \check{\mathcal{Q}}_2{}^i$ and $\sum_{\omega=1}^{\alpha} T^{\omega}_{\breve{\eta}_1}(\hat{u}) < \sum_{\omega=1}^{\alpha} T^{\omega}_{\breve{\eta}_2}(\hat{u}), \quad \sum_{\lambda=1}^{\beta} F^{\lambda}_{\breve{\eta}_1}(\hat{u}) > \sum_{\lambda=1}^{\beta} F^{\lambda}_{\breve{\eta}_2}(\hat{u}), \ \forall \quad \hat{u} \in \breve{\mathfrak{U}}$.
We denote it by $(\hat{u}, \check{\mathcal{Q}}_1{}^i) \subset (\hat{u}, \check{\mathcal{Q}}_2{}^i) \quad \forall \quad i = 1, 2, 3, \ldots, n$.

Example 3.3. Considering data given in Example 2.2, let $\breve{\eta}^1{}_{rpfs}$ and $\breve{\eta}^2{}_{rpfs}$ be two *rpf*-sets such that

$$\begin{aligned} (\hat{u}, \check{\mathcal{Q}}_1) = \{ &< \check{\mathcal{L}}_1, (0.4, 0.5), (0.003, 0.004) >, < \check{\mathcal{L}}_2, (0.5, 0.4), (0.005, 0.004) >, \\ &< \check{\mathcal{L}}_3, (0.5, 0.2), (0.001, 0.006) >, < \check{\mathcal{L}}_4, (0.3, 0.4), (0.006, 0.004) > \}, \end{aligned}$$

and

$$\begin{aligned} (\hat{u}, \check{\mathcal{Q}}_2) = \{ &< \check{\mathcal{L}}_1, (0.3, 0.005), (0.02, 0.1) >, < \check{\mathcal{L}}_2, (0.45, 0.04), (0.05, 0.34) >, \\ &< \check{\mathcal{L}}_3, (0.48, 0.02), (0.01, 0.6) >, < \check{\mathcal{L}}_4, (0.12, 0.2), (0.6, 0.04) > \} \end{aligned}$$

From definition of refined Pythagorean fuzzy subset, it is clear that $(\hat{u}, \check{\mathcal{Q}}_1) \subset (\hat{u}, \check{\mathcal{Q}}_2)$.

Definition 3.4. Equal *rpf*-sets. Let $\breve{\eta}^1{}_{rpfs} = (\hat{u}, \check{\mathcal{Q}}_1)$ and $\breve{\eta}^2{}_{rpfs} = (\hat{u}, \check{\mathcal{Q}}_2)$ be two *rpf*-sets, then $\breve{\eta}^1{}_{rpfs} = \breve{\eta}^2{}_{rpfs}$, if $\breve{\eta}^1{}_{rpfs} \subseteq \breve{\eta}^2{}_{rpfs}$ and $\breve{\eta}^2{}_{rpfs} \subseteq \breve{\eta}^1{}_{rpfs}$.

Example 3.5. Considering data given in Example 2.2, let $\breve{\eta}^1{}_{rpfs}$ and $\breve{\eta}^2{}_{rpfs}$ be two *rpf*-sets such that

$$\begin{aligned} (\hat{u}, \check{\mathcal{Q}}_1) = \{ &< \check{\mathcal{L}}_1, (0.4, 0.5), (0.003, 0.004) >, < \check{\mathcal{L}}_2, (0.5, 0.4), (0.005, 0.004) >, \\ &< \check{\mathcal{L}}_3, (0.5, 0.2), (0.001, 0.006) >, < \check{\mathcal{L}}_4, (0.3, 0.4), (0.006, 0.004) > \}, \end{aligned} \tag{3.1}$$

and

$$(\hat{u}, \check{\mathcal{Q}}_2) = \{ <\check{\mathcal{L}}_1, (0.4, 0.5), (0.003, 0.004)>, <\check{\mathcal{L}}_2, (0.5, 0.4), (0.005, 0.004)>, <\check{\mathcal{L}}_3, (0.5, 0.2), (0.001, 0.006)>, <\check{\mathcal{L}}_4, (0.3, 0.4), (0.006, 0.004)> \}, \quad 3.2$$

are two *rpf*-sets. Then from equations (3.1) and (3.2), it is clear that $\check{\eta}^1{}_{rpfs} = \check{\eta}^2{}_{rpfs}$.

Definition 3.6. Null *rpf*-set. A *rpf*-set $(\hat{u}, \check{\mathcal{Q}}_1)$ is said to be null *rpf*-set if $\sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_1}(\hat{u}) = 0$ and $\sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_1}(\hat{u}) = 0, \ \forall \ \ \hat{u} \in \check{\mathfrak{U}}$. It is denoted by $(\hat{u}, \check{\mathcal{Q}}_1)_{null}$.

Example 3.7. Assuming data given in Example 2.2, the null *rpf*-set is given as

$$(\hat{u}, \check{\mathcal{Q}}_1) = \{ <\check{\mathcal{L}}_1, (0,0), (0,0)>, <\check{\mathcal{L}}_2, (0,0), (0,0)>, <\check{\mathcal{L}}_3, (0,0), (0,0)>, <\check{\mathcal{L}}_4, (0,0), (0,0)> \}.$$

Definition 3.8. Complement of *rpf*-set. The complement of $(\hat{u}, \check{\mathcal{Q}})$ is denoted by $(\hat{u}, \check{\mathcal{Q}}^c)$ and is defined that if $\sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}^c}(\hat{u}) = \sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\xi}}(\hat{u})$ and $\sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}^c}(\hat{u}) = \sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\xi}}(\hat{u})$.

Remark 3.9. The complement of the family of *rpf*-set is defined as $(\hat{u}, \check{\mathcal{Q}}_i{}^c)$ and can be defined in a way that if $\sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_i^c}(\hat{u}) = \sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_i}(\hat{u})$ and $\sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_i^c}(\hat{u}) = \sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_i}(\hat{u}), \quad i = 1, 2, 3, \ldots, n.$

Example 3.10. Considering data given in Example 2.2, if we have *rpf*-set $(\hat{u}, \check{\mathcal{Q}})$ given as

$$\check{\xi}_{prfs} = \{ <\check{\mathcal{L}}_1, (0.02, 0.1), (0.3, 0.5)>, <\check{\mathcal{L}}_2, (0.05, 0.34), (0.45, 0.04)>, <\check{\mathcal{L}}_3, (0.01, 0.6), (0.3, 0.02)>, <\check{\mathcal{L}}_4, (0.6, 0.04), (0.12, 0.2)> \},$$

then the complement of *rpf*-set $(\hat{u}, \check{\mathcal{Q}})$ is given as

$$\check{\eta}^c{}_{rpf-set} = \{ <\check{\mathcal{L}}_1, (0.3, 0.5), (0.02, 0.1)>, <\check{\mathcal{L}}_2, (0.45, 0.04), (0.05, 0.34)>, <\check{\mathcal{L}}_3, (0.3, 0.02), (0.01, 0.6)>, <\check{\mathcal{L}}_4, (0.12, 0.2), (0.6, 0.04)> \}.$$

4. Aggregation Operators of *rpf*-set

This portion describes the set-theoretic operations of *rpf*-set by utilizing the data presented in Example (2.2).

Definition 4.1. Union of two *rpf*-sets. The union of two *rpf*-sets $(\hat{u}, \check{\mathcal{Q}}_1)$ and $(\hat{u}, \check{\mathcal{Q}}_2)$ is denoted by $(\hat{u}, \check{\mathcal{Q}}_1) \cup (\hat{u}, \check{\mathcal{Q}}_2)$ and it is defined as $(\hat{u}, \check{\mathcal{Q}}_1) \cup (\hat{u}, \check{\mathcal{Q}}_2) = (\hat{u}, \Upsilon)$, where $\Upsilon = \check{\mathcal{Q}}_1 \cup \check{\mathcal{Q}}_2$, and truth and false membership of $(\hat{u}, \Upsilon)$ is defined in such a way that

$$T_{\Upsilon}(\hat{u}) = \max\Big(\sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_1}(\hat{u}), \sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_2}(\hat{u})\Big)$$

$$F_{\Upsilon}(\hat{u}) = \min\Big(\sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_1}(\hat{u}), \sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big)$$

Remark 4.2. The union of the two family of *rpf*-sets $(\hat{u}, \check{\mathcal{Q}}_1{}^i)$ and $(\hat{u}, \check{\mathcal{Q}}_2{}^i)$ is denoted by $(\hat{u}, \check{\mathcal{Q}}_1{}^i) \cup (\hat{u}, \check{\mathcal{Q}}_2{}^i)$ and it is defined as $(\hat{u}, \check{\mathcal{Q}}_1{}^i) \cup (\hat{u}, \check{\mathcal{Q}}_2{}^i) = (\hat{u}, \Upsilon^i), \quad \Upsilon^i = \check{\mathcal{Q}}_1{}^i \cup \check{\mathcal{Q}}_2{}^i, \ i = 1, 2, 3, \ldots, n,$ and truth and false membership of $(\hat{u}, \Upsilon^i)$ is defined in such a way that

$$T_{\Upsilon^i}(\hat{u}) = \max\Big(\sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_1}(\hat{u}), \sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_2}(\hat{u})\Big),$$

$$F_{\Upsilon^i}(\hat{u}) = \min\Big(\sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_1}(\hat{u}), \sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big).$$

Example 4.3. Assuming data given in Example 2.2, let

$$(\hat{u}, \check{\mathcal{Q}}_1) = \{ <\check{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\check{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)>, <\check{\mathcal{L}}_3, (0.001, 0.006), (0.5, 0.2)>, <\check{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)> \},$$

and

$$(\hat{u}, \check{\mathcal{Q}}_2) = \{ <\check{\mathcal{L}}_1, (0.02, 0.1), (0.3, 0.005)>, <\check{\mathcal{L}}_2, (0.05, 0.34), (0.45, 0.04)>, <\check{\mathcal{L}}_3, (0.01, 0.6), (0.5, 0.02)>, <\check{\mathcal{L}}_4, (0.6, 0.04), (0.12, 0.2)> \}$$

be two *rpf*-set. Then the union of $(\hat{u}, \check{\mathcal{Q}}_1)$ and $(\hat{u}, \check{\mathcal{Q}}_2)$ is given as

$$(\hat{u}, \Upsilon) = \{ <\check{\mathcal{L}}_1, (0.02, 0.1), (0.3, 0.005)>, <\check{\mathcal{L}}_2, (0.05, 0.34), (0.45, 0.04)>, <\check{\mathcal{L}}_3, (0.01, 0.6), (0.5, 0.02)>, <\check{\mathcal{L}}_4, (0.6, 0.04), (0.12, 0.2)> \}$$

Definition 4.4. Intersection of two *rpf*-sets The intersection of two *rpf*-sets $(\hat{u}, \check{\mathcal{Q}}_1)$ and $(\hat{u}, \check{\mathcal{Q}}_2)$ is denoted by $(\hat{u}, \check{\mathcal{Q}}_1) \cap (\hat{u}, \check{\mathcal{Q}}_2)$ and it is defined as $(\hat{u}, \check{\mathcal{Q}}_1) \cap (\hat{u}, \check{\mathcal{Q}}_2) = (\hat{u}, \Upsilon)$, where $\Upsilon = \check{\mathcal{Q}}_1 \cap \check{\mathcal{Q}}_2$, and truth and false membership of $(\hat{u}, \Upsilon)$ is defined in such a way that

$$T_{\Upsilon}(\hat{u}) = \min\Big(\sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_1}(\hat{u}), \sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_2}(\hat{u})\Big),$$

$$F_{\Upsilon}(\hat{u}) = \max\Big(\sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_1}(\hat{u}), \sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big).$$

Remark 4.5. The intersection of the two family of *rpf*-sets $(\hat{u}, \check{\mathcal{Q}}_1{}^i)$ and $(\hat{u}, \check{\mathcal{Q}}_2{}^i)$ is denoted by $(\hat{u}, \check{\mathcal{Q}}_1{}^i) \cap (\hat{u}, \check{\mathcal{Q}}_2{}^i)$ and it is defined as $(\hat{u}, \check{\mathcal{Q}}_1{}^i) \cap (\hat{u}, \check{\mathcal{Q}}_2{}^i) = (\hat{u}, \Upsilon^i), \quad \Upsilon^i = \check{\mathcal{Q}}_1{}^i \cap \check{\mathcal{Q}}_2{}^i, \ i = 1, 2, 3, \ldots, n,$ and truth and false membership of $(\hat{u}, \Upsilon^i)$ is defined in such a way that

$$T_{\Upsilon^k}(\hat{u}) = \min\Big(\sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_1}(\hat{u}), \sum_{\omega=1}^{\alpha} T^{\omega}_{\check{\eta}_2}(\hat{u})\Big),$$

$$F_{\Upsilon^i}(\hat{u}) = \max\Big(\sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_1}(\hat{u}), \sum_{\lambda=1}^{\beta} F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big).$$

Example 4.6. Considering data given in Example 2.2, let

$$(\hat{u},\check{\mathcal{Q}}_1)=\{<\check{\mathcal{L}}_1,(0.003,0.004),(0.4,0.5)>,<\check{\mathcal{L}}_2,(0.005,0.004),(0.5,0.4)>,\\ <\check{\mathcal{L}}_3,(0.001,0.006),(0.5,0.2)>,<\check{\mathcal{L}}_4,(0.006,0.004),(0.3,0.4)>\},$$

and

$$(\hat{u},\check{\mathcal{Q}}_2)=\{<\check{\mathcal{L}}_1,(0.02,0.1),(0.3,0.005)>,<\check{\mathcal{L}}_2,(0.05,0.34),(0.45,0.04)>,\\ <\check{\mathcal{L}}_3,(0.01,0.6),(0.5,0.02)>,<\check{\mathcal{L}}_4,(0.6,0.04),(0.12,0.2)>\}$$

be two *rpf*-sets. Then the intersection of $(\hat{u},\check{\mathcal{Q}}_1)$ and $(\hat{u},\check{\mathcal{Q}}_2)$ is given as

$$(\hat{u},\check{\Upsilon})=\{<\check{\mathcal{L}}_1,(0.003,0.004),(0.4,0.5)>,<\check{\mathcal{L}}_2,(0.005,0.004),(0.5,0.4)>,\\ <\check{\mathcal{L}}_3,(0.001,0.006),(0.5,0.2)>,<\check{\mathcal{L}}_4,(0.006,0.004),(0.3,0.4)>\},$$

Definition 4.7. Extended intersection of two *rpf*-sets. The intersection of two *rpf*-sets $(\hat{u},\check{\mathcal{Q}}_1)$ and $(\hat{u},\check{\mathcal{Q}}_2)$ is denoted by $(\hat{u},\check{\mathcal{Q}}_1)\cap_\varepsilon(\hat{u},\check{\mathcal{Q}}_2)$ and it is defined as $(\hat{u},\check{\mathcal{Q}}_1)\cap_\varepsilon(\hat{u},\check{\mathcal{Q}}_2)=(\hat{u},\Upsilon)$, where $\Upsilon=\check{\mathcal{Q}}_1\cup\check{\mathcal{Q}}_2$, and truth and false membership of $(\hat{u},\Upsilon)$ is defined in such a way that

$$T_\Upsilon(\hat{u})=\min\Big(\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_1}(\hat{u}),\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_2}(\hat{u})\Big),$$

$$F_\Upsilon(\hat{u})=\max\Big(\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_1}(\hat{u}),\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big).$$

Remark 4.8. The intersection of the two family of *rpf*-sets $(\hat{u},\check{\mathcal{Q}}_1{}^i)$ and $(\hat{u},\check{\mathcal{Q}}_2{}^i)$ is denoted by $(\hat{u},\check{\mathcal{Q}}_1{}^i)\cap_\varepsilon(\hat{u},\check{\mathcal{Q}}_2{}^i)$ and it is defined as $(\hat{u},\check{\mathcal{Q}}_1{}^i)\cap_\varepsilon(\hat{u},\check{\mathcal{Q}}_2{}^i)=(\hat{u},\Upsilon^i)$, $\Upsilon^i=\check{\mathcal{Q}}_1{}^i\cup\check{\mathcal{Q}}_2{}^i$, $i=1,2,3,\ldots,n$, and truth and false membership of $(\hat{u},\Upsilon^i)$ is defined in such a way that

$$T_{\Upsilon^i}(\hat{u})=\min\Big(\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_1}(\hat{u}),\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_2}(\hat{u})\Big),$$

$$F_{\Upsilon^i}(\hat{u})=\max\Big(\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_1}(\hat{u}),\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big).$$

Example 4.9. Assuming data given in Example 2.2, suppose that

$$(\hat{u},\check{\mathcal{Q}}_1)=\{<\check{\mathcal{L}}_1,(0.003,0.004),(0.4,0.5)>,<\check{\mathcal{L}}_2,(0.005,0.004),(0.5,0.4)>,\\ <\check{\mathcal{L}}_3,(0.001,0.006),(0.5,0.2)>\},$$

and

$$(\hat{u},\check{\mathcal{Q}}_2)=\{<\check{\mathcal{L}}_3,(0.01,0.6),(0.5,0.02)>,<\check{\mathcal{L}}_4,(0.6,0.04),(0.12,0.2)>\}$$

be two *rpf*-sets. Then the extended intersection of $(\hat{u},\check{\mathcal{Q}}_1)$ and $(\hat{u},\check{\mathcal{Q}}_2)$ is given as

$$(\hat{u},\check{\Upsilon})=\{<\check{\mathcal{L}}_1,(0.003,0.004),(0.4,0.5)>,<\check{\mathcal{L}}_2,(0.005,0.004),(0.5,0.4)>,\\ <\check{\mathcal{L}}_3,(0.001,0.006),(0.5,0.2)>,<\check{\mathcal{L}}_4,(0.6,0.04),(0.12,0.2)>\}.$$

Definition 4.10. Restricted union of two *rpf*-sets. The restricted union of two *rpf*-sets $(\hat{u},\check{\mathcal{Q}}_1)$ and $(\hat{u},\check{\mathcal{Q}}_2)$ is denoted by $(\hat{u},\check{\mathcal{Q}}_1)\cup_R(\hat{u},\check{\mathcal{Q}}_2)$ and it is defined as $(\hat{u},\check{\mathcal{Q}}_1)\cup_R(\hat{u},\check{\mathcal{Q}}_2)=(\hat{u},\Upsilon)$, where $\Upsilon=\check{\mathcal{Q}}_1\cap_R\check{\mathcal{Q}}_2$, and truth and false membership of $(\hat{u},\Upsilon)$ is defined in such a way that

$$T_\Upsilon(\hat{u})=\max\Big(\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_1}(\hat{u}),\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_2}(\hat{u})\Big),$$

$$F_\Upsilon(\hat{u})=\min\Big(\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_1}(\hat{u}),\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big).$$

Remark 4.11. The restricted union of the two family of *rpf*-sets $(\hat{u},\check{\mathcal{Q}}_1{}^i)$ and $(\hat{u},\check{\mathcal{Q}}_2{}^i)$ is denoted by $(\hat{u},\check{\mathcal{Q}}_1{}^i)\cup_R(\hat{u},\check{\mathcal{Q}}_2{}^i)$ and it is defined as $(\hat{u},\check{\mathcal{Q}}_1{}^i)\cup_R(\hat{u},\check{\mathcal{Q}}_2{}^i)=(\hat{u},\Upsilon^i)$, $\Upsilon^i=\check{\mathcal{Q}}_1{}^i\cap_R\check{\mathcal{Q}}_2{}^i$, $i=1,2,3,\ldots,n$, and truth and false membership of $(\hat{u},\Upsilon^i)$ is defined in such a way that

$$T_{\Upsilon^i}(\hat{u})=\max\Big(\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_1}(\hat{u}),\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_2}(\hat{u})\Big),$$

$$F_{\Upsilon^i}(\hat{u})=\min\Big(\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_1}(\hat{u}),\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big).$$

Example 4.12. Considering data given in Example 2.2, let

$$(\hat{u},\check{\mathcal{Q}}_1)=\{<\check{\mathcal{L}}_1,(0.003,0.004),(0.4,0.5)>,<\check{\mathcal{L}}_2,(0.005,0.004),(0.5,0.4)>,\\ <\check{\mathcal{L}}_3,(0.001,0.006),(0.5,0.2)>\},$$

and

$$(\hat{u},\check{\mathcal{Q}}_2)=\{<\check{\mathcal{L}}_3,(0.01,0.6),(0.5,0.02)>,<\check{\mathcal{L}}_4,(0.6,0.04),(0.12,0.2)>\}$$

be two *rpf*-sets. Then the restricted union of $(\hat{u},\check{\mathcal{Q}}_1)$ and $(\hat{u},\check{\mathcal{Q}}_2)$ is given as

$$(\hat{u},\check{\Upsilon})=\{<\check{\mathcal{L}}_3,(0.01,0.06),(0.5,0.02)>\}.$$

Definition 4.13. Restricted intersection of two *rpf*-sets. The restricted intersection of two *rpf*-sets $(\hat{u},\check{\mathcal{Q}}_1)$ and $(\hat{u},\check{\mathcal{Q}}_2)$ is denoted by $(\hat{u},\check{\mathcal{Q}}_1)\cap_R(\hat{u},\check{\mathcal{Q}}_2)$ and it is defined as $(\hat{u},\check{\mathcal{Q}}_1)\cap_R(\hat{u},\check{\mathcal{Q}}_2)=(\hat{u},\Upsilon)$, where $\Upsilon=\check{\mathcal{Q}}_1\cap_R\check{\mathcal{Q}}_2$, and truth and false membership of $(\hat{u},\Upsilon)$ is defined in such a way that

$$T_\Upsilon(\hat{u})=\min\Big(\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_1}(\hat{u}),\sum_{\omega=1}^{\alpha}T^{\omega}_{\check{\eta}_2}(\hat{u})\Big)$$

$$F_\Upsilon(\hat{u})=\max\Big(\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_1}(\hat{u}),\sum_{\lambda=1}^{\beta}F^{\lambda}_{\check{\eta}_2}(\hat{u})\Big).$$

Remark 4.14. The restricted intersection of the two family of *rpf*-sets $(\hat{u},\check{\mathcal{Q}}_1{}^i)$ and $(\hat{u},\check{\mathcal{Q}}_2{}^i)$ is denoted by $(\hat{u},\check{\mathcal{Q}}_1{}^i)\cap_R(\hat{u},\check{\mathcal{Q}}_2{}^i)$ and it is defined as $(\hat{u},\check{\mathcal{Q}}_1{}^i)\cap_R(\hat{u},\check{\mathcal{Q}}_2{}^i)=(\hat{u},\Upsilon^i)$, $\Upsilon^i=\check{\mathcal{Q}}_1{}^i\cap_R\check{\mathcal{Q}}_2{}^i$, $i=1,2,3,\ldots,n$, and truth and false membership of $(\hat{u},\Upsilon)$ is defined in such a way that

$$T_{\Upsilon^i}(\hat{u}) = \min\big(\sum_{\omega=1}^{\alpha} T^{\omega}_{\breve{\eta}_1}(\hat{u}), \sum_{\omega=1}^{\alpha} T^{\omega}_{\breve{\eta}_2}(\hat{u})\big),$$

$$F_{\Upsilon^i}(\hat{u}) = \max\big(\sum_{\lambda=1}^{\beta} F^{\lambda}_{\breve{\eta}_1}(\hat{u}), \sum_{\lambda=1}^{\beta} F^{\lambda}_{\breve{\eta}_2}(\hat{u})\big).$$

Example 4.15. Considering data given in Example 2.2, let

$$(\hat{u}, \breve{\mathcal{Q}}_1) = \{ <\breve{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)> \},$$

and
$$(\hat{u}, \breve{\mathcal{Q}}_2) = \{ <\breve{\mathcal{L}}_2, (0.05, 0.34), (0.45, 0.04)>, <\breve{\mathcal{L}}_4, (0.6, 0.04), (0.12, 0.2)> \},$$

be two *rpf*-sets. Then the restricted intersection of $(\hat{u}, \breve{\mathcal{Q}}_1)$ and $(\hat{u}, \breve{\mathcal{Q}}_2)$ is given as

$$(\hat{u}, \breve{\Upsilon}) = \{ <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)> \}.$$

Definition 4.16. Restricted difference of two *rpf*-sets. The restricted difference of two *rpf*-sets $(\hat{u}, \breve{\mathcal{Q}}_1)$ and $(\hat{u}, \breve{\mathcal{Q}}_2)$ is denoted by $(\hat{u}, \breve{\mathcal{Q}}_1) -_R (\hat{u}, \breve{\mathcal{Q}}_2)$ and it is defined as $(\hat{u}, \breve{\mathcal{Q}}_1) -_R (\hat{u}, \breve{\mathcal{Q}}_2) = (\hat{u}, \Upsilon)$, where $\Upsilon = \breve{\mathcal{Q}}_1 -_R \breve{\mathcal{Q}}_2$.

Example 4.17. Considering data given in Example 2.2, let

$$(\hat{u}, \breve{\mathcal{Q}}_1) = \{ <\breve{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)>, <\breve{\mathcal{L}}_3, (0.001, 0.006), (0.5, 0.2)>, <\breve{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)> \},$$

and

$$(\hat{u}, \breve{\mathcal{Q}}_2) = \{ <\breve{\mathcal{L}}_1, (0.02, 0.1), (0.3, 0.005)>, <\breve{\mathcal{L}}_2, (0.05, 0.34), (0.45, 0.04)>, <\breve{\mathcal{L}}_3, (0.01, 0.6), (0.5, 0.02)> \},$$

be two *rpf*-sets. Then the restricted difference operation is given as

$$(\Upsilon, S) = \{<\breve{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)>\}.$$

Proposition 4.18.

1. $(\hat{u}, \breve{\mathcal{Q}})\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}) = (\hat{u}, \breve{\mathcal{Q}}) = (\hat{u}, \breve{\mathcal{Q}})\breve{\cup}_R(\hat{u}, \breve{\mathcal{Q}})$
2. $(\hat{u}, \breve{\mathcal{Q}})\breve{\cap}((\hat{u}, \breve{\mathcal{Q}}) = (\hat{u}, \breve{\mathcal{Q}}) = (\hat{u}, \breve{\mathcal{Q}})\breve{\cap}_\varepsilon(\hat{u}, \breve{\mathcal{Q}})$
3. $(\hat{u}, \breve{\mathcal{Q}})\breve{\cup}\Phi = (\hat{u}, \breve{\mathcal{Q}}) = (\hat{u}, \breve{\mathcal{Q}})\breve{\cup}_R\Phi$
4. $(\hat{u}, \breve{\mathcal{Q}})\breve{\cap}\breve{\mathfrak{U}} = (\hat{u}, \breve{\mathcal{Q}}) = (\Omega, \breve{\mathcal{Q}})\breve{\cap}_\varepsilon\breve{\mathfrak{U}}$
5. $(\hat{u}, \breve{\mathcal{Q}})\breve{\cup}\breve{\mathfrak{U}} = \breve{\mathfrak{U}} = (\hat{u}, \breve{\mathcal{Q}})\breve{\cup}_R\breve{\mathfrak{U}}$
6. $(\hat{u}, \breve{\mathcal{Q}})\breve{\cap}\breve{\Phi} = \breve{\Phi} = (\hat{u}, \breve{\mathcal{Q}})\breve{\cap}_\varepsilon\breve{\Phi}$
7. $((\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}_2))^c = (\hat{u}, \breve{\mathcal{Q}}_1)^c \cap_\varepsilon (\hat{u}, \breve{\mathcal{Q}}_2)^c$
8. $((\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cap}_\varepsilon(\hat{u}, \breve{\mathcal{Q}}_2))^c = (\hat{u}, \breve{\mathcal{Q}}_1)^c\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}_2)^c$
9. $((\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}_2)) = (\hat{u}, \breve{\mathcal{Q}}_2))\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}_1))$
10. $((\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cup}_R(\hat{u}, \breve{\mathcal{Q}}_2)) = (\hat{u}, \breve{\mathcal{Q}}_2))\breve{\cup}_R(\hat{u}, \breve{\mathcal{Q}}_1))$
11. $((\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cap}(\hat{u}, \breve{\mathcal{Q}}_2)) = (\hat{u}, \breve{\mathcal{Q}}_2))\breve{\cap}(\hat{u}, \breve{\mathcal{Q}}_1))$
12. $((\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cap}_\varepsilon(\hat{u}, \breve{\mathcal{Q}}_2)) = (\hat{u}, \breve{\mathcal{Q}}_2))\breve{\cup}_\varepsilon(\hat{u}, \breve{\mathcal{Q}}_1))$

Example 4.19. For (1), assuming data from Example (2.2), let

$$(\hat{u}, \breve{\mathcal{Q}}) = \{ <\breve{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)>, <\breve{\mathcal{L}}_3, (0.001, 0.006), (0.5, 0.2)>, <\breve{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)> \},$$

It can be observed that

$$(\hat{u}, \breve{\mathcal{Q}})\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}) = \{ <\breve{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)>, <\breve{\mathcal{L}}_3, (0.001, 0.006), (0.5, 0.2)>, <\breve{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)> \},$$

$= (\hat{u}, \breve{\mathcal{Q}}) = (\hat{u}, \breve{\mathcal{Q}})\breve{\cup}_R(\hat{u}, \breve{\mathcal{Q}})$.
Similarly, we can prove (2).
For (3), let

$$(\hat{u}, \breve{\mathcal{Q}}) = \{ <\breve{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)>, <\breve{\mathcal{L}}_3, (0.001, 0.006), (0.5, 0.2)>, <\breve{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)> \},$$

It can be observed that

$$(\hat{u}, \breve{\mathcal{Q}})\breve{\cup}\Phi = \{ <\breve{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)>, <\breve{\mathcal{L}}_3, (0.001, 0.006), (0.5, 0.2)>, <\breve{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)> \},$$

$= (\hat{u}, \breve{\mathcal{Q}}) = (\hat{u}, \breve{\mathcal{Q}})\breve{\cup}_R\Phi$.
Similarly, we can prove (4). For (5), let

$$(\hat{u}, \breve{\mathcal{Q}}) = \{ <\breve{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)>, <\breve{\mathcal{L}}_3, (0.001, 0.006), (0.5, 0.2)>, <\breve{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)> \},$$

It can be observed that

$$(\hat{u}, \breve{\mathcal{Q}})\breve{\cup}\breve{\mathfrak{U}} = \{ <\breve{\mathcal{L}}_1, (0.003, 0.004), (0.4, 0.5)>, <\breve{\mathcal{L}}_2, (0.005, 0.004), (0.5, 0.4)>, <\breve{\mathcal{L}}_3, (0.001, 0.006), (0.5, 0.2)>, <\breve{\mathcal{L}}_4, (0.006, 0.004), (0.3, 0.4)> \} \cup U,$$

$(\hat{u}, \breve{\mathcal{Q}})\breve{\cup}\breve{\mathfrak{U}} = U = (\hat{u}, \breve{\mathcal{Q}})\breve{\cup}_R\breve{\mathfrak{U}}$.
Similarly, we can prove (6).
For (7), consider $L.H.S$

$$(\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}_2) = \{ <\breve{\mathcal{L}}_1, (0.02, 0.1), (0.3, 0.5)>, <\breve{\mathcal{L}}_2, (0.05, 0.34), (0.45, 0.04)>, <\breve{\mathcal{L}}_3, (0.01, 0.6), (0.5, 0.02)>, <\breve{\mathcal{L}}_4, (0.6, 0.04), (0.12, 0.2)> \}$$

Then

$$((\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}_2))^c = \{ <\breve{\mathcal{L}}_1, (0.3, 0.5), (0.02, 0.1)>, <\breve{\mathcal{L}}_2, (0.45, 0.04), (0.05, 0.34))>, <\breve{\mathcal{L}}_3, (0.5, 0.02), (0.01, 0.6)>, <\breve{\mathcal{L}}_4, (0.12, 0.2), (0.6, 0.04)> \}$$

Now, consider $R.H.S$

$$(\hat{u}, \breve{\mathcal{Q}}_1)^c\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}_2)^c = \{ <\breve{\mathcal{L}}_1, (0.3, 0.5), (0.02, 0.1)>, <\breve{\mathcal{L}}_2, (0.45, 0.04), (0.05, 0.34))>, <\breve{\mathcal{L}}_3, (0.5, 0.02), (0.01, 0.6)>, <\breve{\mathcal{L}}_4, (0.12, 0.2), (0.6, 0.04)> \}$$

From this, it is clear that $L.H.S = R.H.S$
For (9), let
$L.H.S$:

$$((\hat{u}, \breve{\mathcal{Q}}_1)\breve{\cup}(\hat{u}, \breve{\mathcal{Q}}_2)) = \{ <\breve{\mathcal{L}}_1, (0.02, 0.1), (0.3, 0.5)>, <\breve{\mathcal{L}}_2, (0.05, 0.34), (0.45, 0.04)>, <\breve{\mathcal{L}}_3, (0.01, 0.6), (0.5, 0.02)>, <\breve{\mathcal{L}}_4, (0.6, 0.04), (0.12, 0.2)> \} \tag{4.1}$$

R.H.S:

$$((\hat{u},\breve{\mathcal{Q}}_2)\breve{\cup}(\hat{u},\breve{\mathcal{Q}}_1)) = \{ <\breve{\mathcal{L}}_1,(0.02,0.1),(0.3,0.5)>,<\breve{\mathcal{L}}_2,(0.05,0.34),(0.45,0.04)>, <\breve{\mathcal{L}}_3,(0.01,0.6),(0.5,0.02)>,<\breve{\mathcal{L}}_4,(0.6,0.04),(0.12,0.2)> \} \tag{4.2}$$

Similarly, other laws can easily be proved.

5. Conclusion

In this paper, a conceptual context of *rpf*-set is devised to tackle the scenario with entitlement of sub-grades with respect to membership and non-membership degrees of existing fuzzy set-like models. Some essential properties, aggregation operations, and set-theoretic laws are characterized by employing theoretic and analytical approaches. The future scope of the proposed study is broad in the sense that certain kinds of similarity and distance measures, entropies, and aggregations can be calculated between *rpf*-sets using this context.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

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