Received: 3 December 2021 | Revised: 21 January 2022 | Accepted: 31 January 2022 | Published online: 1 February 2022

RESEARCH ARTICLE

Journal of Computational and Cognitive Engineering
2022, Vol. 1(1) 37–41
DOI: 10.47852/bonviewJCCE6152118205514

BON VIEW PUBLISHING

Multiset Modules

Sunil Jacob John[1], **Suma P**[1] **and Athira T. M.**[1,*]

[1]*Department of Mathematics, National Institute of Technology Calicut, India*

Abstract: Multiset modules and their properties are introduced in this paper. Some interesting properties are obtained, such as the countable intersection of multiset modules is multiset module, but the union need not be. Also, the sub-multiset module is defined and illustrated with suitable examples. Homomorphism and isomorphism in the contest of multisets are defined, and some valuable theorems are proved. Then the quotient module is proposed, and the relation that $\mathcal{M}/\ker f$ is isomorphic to $\mathrm{Im}\, f$ for a multiset homomorphism *f*. Multiset modules drawn from a $\mathbb{Z}$ module are of particular interest and proved that if $\mathcal{L} \in ML[\mathbb{Z}\mathcal{M}]$, then $\mathcal{L}$ is an mset group under addition, and conversely, every mset abelian group drawn from $\mathbb{Z}$ is an element of $ML[\mathbb{Z}\mathcal{M}]$.

Keywords: multiset module, multiset homomorphism, multiset isomorphism

1. Introduction

George Cantor's formal definition of a set is as a well-defined collection of distinct objects. The terms well defined and distinct in this, and their integrity, or lack of after that, eventually led to the formation of generalized set theory, mainly consisting of fuzzy sets, multisets, rough sets, soft sets, etc. Out of these, multiset is the primary focus here. The distinctness property is violated in multisets such that duplicate elements can occur in it. Since multiplicity of elements is the key feature of a multiset, all the findings are based on the count value. In other words, the characteristic function plays a significant role in this work.

Multisets, just like sets, are associative containers but differ because there is a possibility for the same value to be assigned to multiple elements. A crisp set can distinguish members and nonmembers of the universal set with the help of characteristic function, which gives value 1 for members and 0 for nonmembers of the given collection. The membership function is further broadened to ensure that the values assigned to the members of the universal set fall over the range of positive integers, and such a defined set is known as the multiset. As mentioned earlier, crisp sets take membership values 0 or 1, but for the case of fuzzy sets (Zadeh, 1965), membership function ranges in [0, 1]. If we move on to multisets, the values can be positive integers. The fuzzy set theory and its results can be applied in many areas, especially computer science. Multisets are also emerging in various domains nowadays, although they are broadly uninvestigated. A significant example would be JAVA's other version, called GUAVA, developed by Google, which is chiefly based on multisets and their properties as opposed to fuzzy sets. During the process of information retrieval, duplicates may arrive at various phases. In such a context, the need for multisets and multiset operations arises. For example, in a cyber-investigation, hitting on a particular website and phone number in a tower at some time interval is a situation where multisets are more suitable than other forms of sets.

Algebra and algebraic structures such as group, ring, etc., are significant in Mathematics. These structures are based on classical set theory. Questions such as why classical sets can be replaced by multisets, the effects of such a change, and what would happen to the sentences and results if such a change occurred led to research into it. Fundamental works related to multisets and its extensions can be seen in Blizard (1989) and Blizard (1991), Wildberger (2003), Radoaca (2015), and Shravan and Tripathy (2019). An overview of various applications of multisets is given in Singh et al. (2007).

Algebraic structures built on fuzzy sets are useful in disciplines, particularly Chemistry, Physics, Computer Science. These fuzzy-set-based structures have been discussed and deliberated in recent years. The basic notions of fuzzy groups, as well as fuzzy groupoids, have been explained by Rosenfeld (1971). Rings, prime ideals, maximal ideals, left and right ideals, for a general ring R in the context of fuzzy sets are explained by Sebastian et al. (2012). Li et al. (2013) have given a remarkable type of (λ, μ) fuzzy subgroup. Even though these reside in multisets, their applications and possibilities have persistently unexplored. Some of them, such as multigroups and their results, are studied in Ibrahim and Ejegwa (2016), Tripathy et al. (2018), Ejegwa (2017), Nazmul et al. (2013), Tella and Daniel (2013), and Awolola and Ibrahim (2016). Rajarajeswari and Uma (2013) have investigated various aspects and applications of fuzzy multisets along with Sebastian and Ramakrishnan (2011).

This paper explores the concept of modules in a multiset context, and different results are presented. In Section 2, different fundamental definitions and properties related to multisets, multiset groups, and multiset rings are included. Section 3 is devoted to the extensive study of multiset modules. Mset module homomorphism and isomorphism are studied. Mset modules drawn from a Z module are considered with particular interest. Section 4 gives conclusions and future directions of research.

*Corresponding author: Athira T. M., Department of Mathematics, National Institute of Technology Calicut, India. Email: athiratm999maths@gmail.com

2. Preliminaries and Basic Definitions

This section explains basic definitions of multisets and multiset operations and multiset groups and rings.

Definition 2.1. (Knuth, 2014) Let S be a nonempty set. A multiset $\mathcal{M}$ taken from S is given by a function $\mathcal{C}_{\mathcal{M}}$ defined as $\mathcal{C}_{\mathcal{M}} : S \to \mathbb{N} \cup \{0\}$. For each $s \in S, \mathcal{C}_{\mathcal{M}}(s)$ is the characteristic value of s in $\mathcal{M}$ and denotes the number of occurrences of s in $\mathcal{M}$.

Let $\mathcal{M}$ be an mset from S with s_1 coming k_1 times, s_2 coming k_2 times, and so on s_n coming k_n times. Then $\mathcal{M}$ is denoted as $\mathcal{M} = \{k_1|s_1, k_2|s_2, \cdots, k_n|s_n\}$.

A multiset $\mathcal{M}$ is empty if $\mathcal{C}_{\mathcal{M}}(s) = 0, \forall s \in S. s \in^n \mathcal{M}$ denotes the element s appearing n times in a multiset $\mathcal{M}$. The term multiset is often shortened to mset.

Definition 2.2. (Girish & John, 2009) Let $\mathcal{M}$ be a multiset taken from a set S. The root set of $\mathcal{M}$ is defined as,

$$\mathcal{M}^* = \{s \in S : \mathcal{C}_{\mathcal{M}}(s) > 0\}.$$

Definition 2.3. (Girish & John, 2009) The multiset space S^n is the set of all multisets whose elements are in S so as no element in the multiset takes place more than n times.

Operations on multisets: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two msets taken from a set S.

(1) **Sub-multiset**: (Yager, 1986) $\mathcal{M}_1$ is a sub multiset of $\mathcal{M}_2(\mathcal{M}_1 \subseteq \mathcal{M}_2)$ if, $\mathcal{C}_{\mathcal{M}_1}(s) \leq \mathcal{C}_{\mathcal{M}_2}(s) \forall s \in S$.

(2) **Equal:** (Yager, 1986) $\mathcal{M}_1$ and $\mathcal{M}_2$ are equal, denoted by $\mathcal{M}_1 - \mathcal{M}_2$, if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\mathcal{M}_2 \subseteq \mathcal{M}_1$.

(3) **Union:** (Knuth, 2014) The $\mathcal{M}_1 \cup \mathcal{M}_2$ is defined by,

$$\mathcal{C}_{\mathcal{M}_1 \cup \mathcal{M}_2}(s) - m\,\{\mathcal{C}_{\mathcal{M}_1}(s), \mathcal{C}_{\mathcal{M}_2}(s)\}, \forall s \in S.$$

(4) **Intersection**: (Knuth, 2014) $\mathcal{M}_1 \cap \mathcal{M}_2$ is defined by,

$$\mathcal{C}_{\mathcal{M}_1 \cap M_2}(s) - m\,\{\mathcal{C}_{\mathcal{M}_1}(s), \mathcal{C}_{\mathcal{M}_2}(s)\}, \forall s \in S.$$

(5) **Addition**: (Knuth, 2014) $\mathcal{M}_1 + \mathcal{M}_2$ is defined as

$$\mathcal{C}_{\mathcal{M}_1 + \mathcal{M}_2}(s) - \mathcal{C}_{\mathcal{M}_1}(s) + \mathcal{C}_{\mathcal{M}_2}(s), \forall s \in S.$$

(6) **Subtraction:** (Knuth, 2014) $\mathcal{M}_1 - \mathcal{M}_2$ is defined as

$$\mathcal{C}_{\mathcal{M}_1 - \mathcal{M}_2}(s) - m\,\{\mathcal{C}_{\mathcal{M}_1}(s) - \mathcal{C}_{\mathcal{M}_2}(s), 0\}, \forall s \in S.$$

(7) **Addition in S^n** : (Girish & John, 2009) Addition of two multisets $\mathcal{M}_1$ and $\mathcal{M}_2$ in S^n can be modified as;

$$\mathcal{C}_{\mathcal{M}_1 + \mathcal{M}_2}(s) - m\,\{n, \mathcal{C}_{\mathcal{M}_1}(s) + \mathcal{C}_{\mathcal{M}_2}(s)\}, \forall s \in S.$$

(8) **Complement**: (Girish & John, 2009) For any multiset $\mathcal{M} \in S^n$, the complement of $\mathcal{M}$, denoted by $\mathcal{M}'$, is given by,

$$\mathcal{C}_{\mathcal{M}'}(s) - n - \mathcal{C}_{\mathcal{M}}(s), \forall s \in S.$$

Example 2.4. The operations of multisets are illustrated in this example. Let S be $\mathbb{N}$ and,
$\mathcal{M}_1 - \{2|5, 3|3, 1|2, 4|1\}, \mathcal{M}_2 - \{1|6, 1|5, 2|4, 4|2, 2 \mid 1\}$ and $\mathcal{M}_3 - \{2|5, 1|3\}$ are msets drawn from S.
(1) $\mathcal{M}_3 \subseteq \mathcal{M}_1$
(2) $\mathcal{M}_1 \cup \mathcal{M}_2 - \{1|6, 2|5, 2|4, 3|3, 4|2, 4|1\}$
(3) $\mathcal{M}_1 \cap \mathcal{M}_2 = \{1|5, 1|2, 2 \mid 1\}$
(4) $\mathcal{M}_1 + 3 - \{2|5, 4|3, 1|2, 4|1\}$
(5) $\mathcal{M}_1 - 3 - \{2|5, 2|3, 1|2, 4|1\}$
(6) $\mathcal{M}_1 + \mathcal{M}_2 - \{1|6, 3|5, 2|4, 3|3, 5|2, 6|1\}$
(7) $\mathcal{M}_1 - \mathcal{M}_2 - \{1|5, 3|3, 2 \mid 1\}$
(8) If we are considering S^{10}, the complement,

$$\mathcal{M}_1' = \{8|5, 7|3, 9/2, 6 \mid 1\}.$$

Multiset groups are msets with their elements taken from a group and the characteristic function of the mset satisfying certain conditions.

Definition 2.5. (Nazmul et al., 2013) Let $(\mathcal{G}, *)$ be a group and $\mathcal{M}$ be an msel taken from $\mathcal{G}$. Then $\mathcal{M}$ is a multiset group (mset group) if,
(1) $\mathcal{C}_{\mathcal{M}}(s * t) \geq \min\{\mathcal{C}_{\mathcal{M}}(s), \mathcal{C}_{\mathcal{M}}(t) : s, t \in \mathcal{G}\}$
(2) $\mathcal{C}_{\mathcal{M}}(s) - \mathcal{C}_{\mathcal{M}}(s^{-1}) \forall s \in \mathcal{G}$ where s^{-1} is the inverse of s in $\mathcal{G}$.

From a ring structure, an mset ring is generated. The two operations used in mset ring are the same as those in the ring from which it is drawn.

Definition 2.6. (Suma & John, 2020) Let $(\mathcal{R}, +, \times)$ be a ring and $\mathcal{M}$ be an mset taken from $\mathcal{R}$. Then $\mathcal{M}$ is a multiset ring (mset ring) if,
(1) $\mathcal{C}_{\mathcal{M}}(s + t) \geq \min\{\mathcal{C}_{\mathcal{M}}(s), \mathcal{C}_{\mathcal{M}}(t) : s, t \in \mathcal{R}\}$
(2) $\mathcal{C}_{\mathcal{M}}(s \times t) \geq \min\{\mathcal{C}_{\mathcal{M}}(s), \mathcal{C}_{\mathcal{M}}(t) : s, l \in \mathcal{R}\}$
(3) $\mathcal{C}_{\mathcal{M}}(-s) - \mathcal{C}_{\mathcal{M}}(s), \forall s \in \mathcal{R}$.

Theorem 2.7. If $\mathcal{M}$ is msel ring drawn from a ring $(\mathcal{R}, +, x)$, then $\mathcal{M}^*$ is a sub ring of $\mathcal{R}$.

Proof. If $s, t \in \mathcal{M}^*$, then $\mathcal{C}_{\mathcal{M}}(s) > 0$ and $\mathcal{C}_{\mathcal{M}}(t) > 0$.

$$\mathcal{C}_{\mathcal{M}}(s - t) = \mathcal{C}_{\mathcal{M}}(s + (-t)) \geq m\{\mathcal{C}_{\mathcal{M}}(s), \mathcal{C}_{\mathcal{M}}(t)\} > 0.$$

Which means $s - t \in \mathcal{M}^*$.

Similarly, $\mathcal{C}_{\mathcal{M}}(s) > 0$ and $\mathcal{C}_{\mathcal{M}}(t) > 0$ will imply $\mathcal{C}_{\mathcal{M}}(s \times t) > 0$. i.e., $s \times t \in \mathcal{M}^*$. So $\mathcal{M}^*$ is a subring of $\mathcal{R}$.

Example 2.8. Let $\mathcal{R} = (\mathbb{Z}_6, +_6, \times_6)$ and $\mathcal{M} = \{3|0, 2|2, 2 \mid 4\}$.

Then $\mathcal{M}$ is an mset ring and $\mathcal{M}^* = \{0, 2, 4\}$ is a subring of $\mathcal{R}$.

3. Multiset Modules

Definition 3.1. Let $\mathcal{R}$ be a commutative ring with unity and $(\mathcal{M}, +, \times)$ be an $\mathcal{R}$-module. A multiset $\mathcal{L}$ drawn from $\mathcal{M}$ is defined as $\mathcal{R}$-multiset module ($\mathcal{R}$-mset module), if
(1) $\mathcal{C}_{\mathcal{L}}(s + t) \geq \min\{\mathcal{C}_{\mathcal{L}}(s), \mathcal{C}_{\mathcal{L}}(t)\} s, t \in \mathcal{M}$.
(2) $\mathcal{C}_{\mathcal{L}}(s) = \mathcal{C}_{\mathcal{L}}(-s)$ for all $s \in \mathcal{M}$ where $-s$ is the additive inverse of s in $\mathcal{M}$.
(3) $\mathcal{C}_{\mathcal{L}}(r \times s) \geq \mathcal{C}_{\mathcal{L}}(s) \forall r \in \mathcal{R}, s \in \mathcal{M}$.

Example 3.2. Let $\mathcal{M} = \mathbb{Z}_6$, which is a $\mathbb{Z}_6$-module under the operations $+_6$ and $\times_6$. Then the mset $\mathcal{L} = \{3|0, 2|2, 2 \mid 4\}$ drawn from $\mathcal{M}$ is an $\mathbb{Z}_6$ mset module.

Notation: $\mathrm{MS}[\mathcal{RM}]$ denotes the set of all msets drawn from an $\mathcal{R}$ module $\mathcal{M}$. $ML[\mathcal{RM}]$ denotes the set of all mset modules drawn from an $\mathcal{R}$-module $\mathcal{M}$.

Proposition 3.3. Let $\mathcal{L} \in ML[\mathcal{RM}]$. Then, the root set $\mathcal{L}^*$ of $\mathcal{L}$ is a submodule of $\mathcal{M}$.

Proof. For any two elements $s, t \in \mathcal{L}^*$ and $r \in \mathcal{R}$, $\mathcal{C}_{\mathcal{L}}(s) > 0$ and $\mathcal{C}_{\mathcal{L}}(t) > 0$.

$$\mathcal{C}_{\mathcal{L}}(s-t) = \mathcal{C}_{\mathcal{L}}(s+(-t) \geq \mathrm{m}\{\mathcal{C}_{\mathcal{L}}(s), \mathcal{C}_{\mathcal{L}}(-t)\}$$
$$= \mathrm{m}\{\mathcal{C}_{\mathcal{L}}(s), \mathcal{C}_{\mathcal{L}}(t)\} > 0.$$

This gives that $s-t \in \mathcal{L}^*$ and so $\mathcal{L}^*$ is a subgroup of $\mathcal{M}$.
Now, $\mathcal{C}_{\mathcal{L}}(r \times s) \geq \mathcal{C}_{\mathcal{L}}(s)$, by condition (3) of definition 3.1.
So, $\mathcal{C}_{\mathcal{L}}(r \times s) > 0$, implies that $r \times s \in \mathcal{L}^*$.
Therefore $\mathcal{L}^*$ is a submodule of $\mathcal{M}$.

Proposition 3.4. Suppose $\mathcal{L} \in ML[\mathcal{RM}]$. Then the level set $\mathcal{L}_r = \{s \in \mathcal{M} : \mathcal{C}_{\mathcal{L}}(s) \geq r\}$, where r is a positive integer, is a submodule of $\mathcal{M}$.

Proof. If $\mathcal{L}_r = \phi$, then it is trivially a submodule of $\mathcal{M}$.

If $\mathcal{L}_r$ has only one element, then it is the identity element of $\mathcal{M}$ and is again a submodule.

Otherwise, let $s, t \in \mathcal{L}_r$ and $s \in \mathcal{R}$. Then $\mathcal{C}_{\mathcal{L}}(s) \geq r$ and $\mathcal{C}_{\mathcal{L}}(t) \geq r$.

Now, $\mathcal{C}_{\mathcal{L}}(s-t) \geq \min\{\mathcal{C}_{\mathcal{L}}(s), \mathcal{C}_{\mathcal{L}}(-t)\}$

$$= \min\{\mathcal{C}_{\mathcal{L}}(s), \mathcal{C}_{\mathcal{L}}(t)\} \geq r$$

Thus $s-t \in \mathcal{L}_r$.
Similarly $\mathcal{C}_{\mathcal{L}}(s \times s) \geq \mathcal{C}_{\mathcal{L}}(s) \geq r \Rightarrow s \times s \in \mathcal{L}_r$.
So $\mathcal{L}_r$ is a submodule of $\mathcal{R}$.

Definition 3.5. Let $\mathcal{K}$ and $\mathcal{L}$ be two mset modules drawn from the same $\mathcal{R}$ - module $\mathcal{M}$. If $\mathcal{K} \subseteq \mathcal{L}$, then $\mathcal{K}$ is said to be sub-mset module of $\mathcal{L}$.

Proposition 3.6. The finite or countable intersection of mset modules is an mset module.

Proof. Let $\mathcal{L}_1, \mathcal{L}_2, \cdots$ are mset modules drawn from an $\mathcal{R}$-module $\mathcal{M}$ and let

$$\mathcal{L} = \cap_j \mathcal{L}_j$$

Take any two elements s and t from $\mathcal{L}$.
Then,
(1) $\mathcal{C}_{\mathcal{L}_j}(s+t) \geq \min\left\{\mathcal{C}_{\mathcal{L}_j}(s), \mathcal{C}_{\mathcal{L}_j}(t)\right\}$ for $j = 1, 2, \cdots$
So,

$$\begin{aligned}\mathcal{C}_{\mathcal{L}}(s+t) &= \min_j \mathcal{C}_{\mathcal{L}_j}(s+t)\\ &\geq \min_j\left\{\mathrm{m}\left\{\mathcal{C}_{\mathcal{L}_j}(s), \mathcal{C}_{\mathcal{L}_j}(t)\right\}\right\}, \text{ by } 1\\ &= \mathrm{m}\left\{\min_j\left(\mathcal{C}_{\mathcal{L}_j}(s)\right), \min_j\left(\mathcal{C}_{\mathcal{L}_j}(t)\right)\right\}\\ &= \mathrm{m}\{\mathcal{C}_{\mathcal{L}}(s), \mathcal{C}_{\mathcal{L}}(t)\}\end{aligned}$$
i.e., $\mathcal{C}_{\mathcal{L}}(s+t) \geq \mathrm{m}\{\mathcal{C}_{\mathcal{L}}(s), \mathcal{C}_{\mathcal{L}}(t)\}$.

Thus we have the first condition of the definition 3.1
Now,

$$\mathcal{C}_{\mathcal{L}}(-s) = \min_j\left\{\mathcal{C}_{\mathcal{L}_j}(-s)\right\} = \mathcal{C}_{\mathcal{L}}(s)$$

which is the second condition of the definition 3.1.

(2) $\mathcal{C}_{\mathcal{L}_j}(r \times s) \geq \mathcal{C}_{\mathcal{L}_j}(s)$

$$\begin{aligned}\mathcal{C}_{\mathcal{L}}(r \times s) &= \min_j \mathcal{C}_{\mathcal{L}_j}(r \times s)\\ &\geq \min_j\left\{\mathcal{C}_{\mathcal{L}_j}(s)\right\},\ by\ (5.2)\\ &= \mathcal{C}_{\mathcal{L}}(s)\end{aligned}$$

This gives the third condition of the definition 3.1 and this completes the proof.

Note: Union of mset modules does not necessarily be an mset module.

Example 3.7. Let $\mathcal{M} = \mathbb{Z}_6$. Then $\mathcal{M}$ is a $\mathbb{Z}_6$-module under $+_6$ and $\times_6$.
Let $\mathcal{L}_1 = \{3|0, 2|2, 2 \mid 4\}$ and $\mathcal{L}_2 = \{2|0, 1|3\}$.
Then $\mathcal{L}_1$ and $\mathcal{L}_2$ belongs to $ML[\mathbb{Z}\mathcal{M}]$.
But $\mathcal{L}_1 \cup \mathcal{L}_2 = \{3|0, 2|2, 1|3, 2|4\} \notin ML[\mathbb{Z}\mathcal{M}]$.

Definition 3.8. Let $\mathcal{K}, \mathcal{L} \in ML[\mathcal{RM}]$. An operation $\oplus$ between $\mathcal{K}$ and $\mathcal{L}$ is defined as
$\mathcal{C}_{\mathcal{K}\oplus\mathcal{L}}(s) = \max\{\min(\mathcal{C}_{\mathcal{K}}(t), \mathcal{C}_{\mathcal{L}}(u)) : s = t+u\}, s, t, u$ are elements of $\mathcal{M}$.

Proposition 3.9. Suppose $\mathcal{K}, \mathcal{L} \in ML[\mathcal{RM}]$. Then $\mathcal{K} \oplus \mathcal{L} \in ML[\mathcal{RM}]$.

Proof. The proof is directly obtained from definition 3.1 and definition 3.8.

Example 3.10. Let $\mathcal{M} = \mathbb{Z}_6$. Then $\mathcal{M}$ is a $\mathbb{Z}_6$-module under $+_6$ and x_6.
Let $\mathcal{L}_1 = \{3|0, 2|2, 24\}$ and $\mathcal{L}_2 = \{2|0, 1|3\}$. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ belongs to $ML[\mathbb{Z}_6\mathcal{M}]$.
$\mathcal{L}_1 \oplus \mathcal{L}_2 = \{2|0, 1|1, 2|2, 1|3, 2|4, 1|5\} \in ML[\mathbb{Z}_6\mathcal{M}]$.

Proposition 3.11. $ML[\mathcal{RM}]$is a semigroup under $\oplus$.

Proof. From proposition, $\oplus$ is a binary operations on $ML[\mathcal{RM}]$. The associative property of $\oplus$ follows from the definition of $\oplus$ and the fact that $\mathcal{M}$ is a module. So, $\oplus$ is an associative binary operation on $ML[\mathcal{RM}]$ and hence, $ML[\mathcal{RM}]$ is a semigroup under $\oplus$.

Definition 3.12. Let $\mathcal{K}$ and $\mathcal{L}$ be two $\mathcal{R}$-mset modules drawn from the $\mathcal{R}$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. An $\mathcal{R}$-module homomorphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ is said to be an $\mathcal{R}$-mset module homomorphism, if $\mathcal{C}_{\mathcal{K}}(s) = \mathcal{C}_{\mathcal{L}}(f(s)), \forall s \in \mathcal{M}_1$.

Definition 3.13. An mset module homomorphism $f : \mathcal{K} \to \mathcal{L}$ is said to be an $\mathcal{R}$-mset module isomorphism, if $f : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism. In this case, we say $\mathcal{K}$ is isomorphic to $\mathcal{L}$ and write it as $\mathcal{K} \cong \mathcal{L}$.

Definition 3.14. Let $\mathcal{K}, \mathcal{L} \in$ ML $[\mathcal{RM}]$ and $f : \mathcal{K} \to \mathcal{L}$ is an $\mathcal{R}$-mset homomorphism. Kernel f (denoted as Ker f) is an mset consisting of those elements s of $\mathcal{M}$ with $f(s) = e$, the identity element of the underlying group of $\mathcal{M}$, with $\mathcal{C}_{\ker f}(s) = \mathcal{C}_{\mathcal{K}}(s)$.

Definition 3.15. Let $\mathcal{K}$ and $\mathcal{L}$ be two mset modules drawn from an $\mathcal{R}$ - module and $f : \mathcal{K} \to \mathcal{L}$ is an $\mathcal{R}$-mset homomorphism. Image f (denoted as Im f) is the mset consisting of those t with $f(s) = t, s \in$ root set of $\mathcal{K}$, and $\mathcal{C}_{Im\, f}(t) = \mathcal{C}_{\mathcal{L}}(t)$.

Theorem 3.16. Let $\mathcal{K}$ and $\mathcal{L}$ be two mset modules drawn from an $\mathcal{R}$ - modules $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively and $f : \mathcal{K} \to \mathcal{L}$ is an $\mathcal{R}$-mset homomorphism. Then Ker f is an sub-mset module of $\mathcal{K}$ and Im f is an sub-mset module of $\mathcal{L}$.

Proof. Let $s, t \in \mathcal{M}_1$ and $r \in \mathcal{R}$. Then by definitions 3.1 and 3.14,

$$\begin{aligned} \mathcal{C}_{Kerf}(s+t) &= \mathcal{C}_{\mathcal{K}}(s+t) \\ &\geq \mathrm{m}\{\mathcal{C}_{\mathcal{K}}(s), \mathcal{C}_{\mathcal{K}}(t)\} \\ &= \mathrm{m}\{\mathcal{C}_{Kerf}(s), \mathcal{C}_{Kerf}(t)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}_{Ker}f(-s) = \mathcal{C}_{\mathcal{K}}(-s) = \mathcal{C}_{\mathcal{K}}(s) &= \mathcal{C}_{Ker}f(s) \\ \mathcal{C}_{Ker}f(r \times s) &= \mathcal{C}_{\mathcal{K}}(r \times s) \\ &\geq \mathcal{C}_{\mathcal{K}}(s) \\ &= \mathcal{C}_{Kerf}(s). \end{aligned}$$

So, we have all the three conditions of definition 3.1 and hence, Ker f is a submodule of $\mathcal{M}_1$.

Now, by putting *Im f* in place of Ker f and mset module $\mathcal{L}$ in the place of $\mathcal{K}$ in the proof, we obtain that Imf is a submodule of $\mathcal{M}_2$.

Theorem 3.17. Let $\mathcal{K}$ and $\mathcal{L}$ be two mset modules drawn from the $\mathcal{R}$ - modules $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively and $f : \mathcal{K} \to \mathcal{L}$ be an $\mathcal{R}$ - mset module homomorphism. Then
(1) $f(A)$ is a sub-mset module of $\mathcal{L}$ for all sub-mset module A of $\mathcal{K}$
(2) $f^{-1}(B)$ is a sub-mset module of $\mathcal{K}$ for all sub-mset module B of $\mathcal{L}$.

Proof. Let A be sub-mset module of $\mathcal{K}$. To show $f(A)$ is an sub-mset module of $\mathcal{L}$, let $a, b \in f(A)$. So there is $c, d \in A$ such that $f(c) = a$ and $f(d) = b$.

$$\begin{aligned} \mathcal{C}_{f(A)}(a+b) &= \mathcal{C}_{f(A)}(f(c)+f(d)) = \mathcal{C}_{f(A)}(f(c+d)) \\ &= \mathcal{C}_{(A)}(c+d) \text{ (By definition 3.12)} \\ &\geq \mathrm{m}\{\mathcal{C}_A(c), \mathcal{C}_A(d)\} \text{ (Since } A \text{ is mset module)} \\ &= \mathrm{m}\{\mathcal{C}_{f(A)}f(c), \mathcal{C}_{f(A)}f(d)\} = \mathrm{m}\{\mathcal{C}_{f(A)}(a), \mathcal{C}_{f(A)}(b)\} \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{C}_{f(A)}(a) = \mathcal{C}_{f(A)}(f(c)) &= \mathcal{C}_A(c) \text{(By definition 3.12)} \\ &= \mathcal{C}_A(-c) \text{(Since } A \text{ is mset module)} \\ &= \mathcal{C}_{f(A)}(f(-c)) = \mathcal{C}_{f(A)}(-a). \end{aligned}$$

For $r \in \mathcal{R}$ and $a \in f(A)$,

$$\begin{aligned} \mathcal{C}_{f(A)}(r \times a) = \mathcal{C}_{f(A)}(r \times f(c)) = \mathcal{C}_{f(A)}f(r \times c) &= \mathcal{C}_A(r \times c) \\ &\geq \mathcal{C}_A(c) \text{ (Since } A \text{ is mset module)} \\ &= \mathcal{C}_{f(A)}(f(c)) = \mathcal{C}_{f(A)}(a). \end{aligned}$$

All the three conditions of definition 3.1 are satisfied for $f(A)$, and thus, it is a sub-mset module of $\mathcal{L}$. Hence, (1) is proved. Similar to the proof of (1), we can prove (2).

Definition 3.18. Let $\mathcal{L} \in MS[\mathcal{RM}]$ and $m \in \mathcal{M}$. The left coset $m\mathcal{L}$ of the mset module $\mathcal{L}$ are the msets characterized by $\mathcal{C}_{m\mathcal{L}}(s) = \mathcal{C}_{\mathcal{L}}(m-s) \forall s \in \mathcal{M}$. The right coset $\mathcal{L}m$ is given by the characteristic value $\mathcal{C}_{\mathcal{L}m}(s) = \mathcal{C}_{\mathcal{L}}(s-m) \forall s \in \mathcal{M}$.

Note: If $\mathcal{L} \in ML[\mathcal{RM}]$, then the left coset is equal to the right coset for a particular $a \in \mathcal{M}$. i.e., $a\mathcal{L} = \mathcal{L}a \forall a \in \mathcal{M}$.

Notation: $\mathcal{M}/\mathcal{L}$ denotes the set of all cosets of an $\mathcal{R}$-mset module $\mathcal{L}$ drawn from an $\mathcal{R}$-module $\mathcal{M}$.

Proposition 3.19. Let $\mathcal{M}$ be an $\mathcal{R}$-module and $\mathcal{L} \in ML[\mathcal{RM}]$. Then $\mathcal{M}/\mathcal{L}$ is a group under the operation

$$\mathcal{C}_{a\mathcal{L} \uplus b\mathcal{L}}(s) = \mathcal{C}_{(a+b)\mathcal{L}}(s), \forall a, b, s \in \mathcal{M}.$$

Proof. If $a\mathcal{L}$ and $b\mathcal{L}$ are two cosets, $a\mathcal{L} \uplus b\mathcal{L}$ is also a coset and thus operation is closed. Since $\mathcal{M}$ is a module, it is associative and so $\mathcal{M}/\mathcal{L}$. $e\mathcal{L}$ is identity element of $\mathcal{M}/\mathcal{L}$ where e is identity element of underlying group $\mathcal{M}$. For $a\mathcal{L} \in \mathcal{M}/\mathcal{L}, (-a)\mathcal{L} \in \mathcal{M}/\mathcal{L}$ is the inverse. Thus $\mathcal{M}/\mathcal{L}$ is a group.

Proposition 3.20. Let $\mathcal{M}$ be an $\mathcal{R}$-module and $\mathcal{L} \in ML[\mathcal{RM}]$. The group $\mathcal{M}/\mathcal{L}$ is an $\mathcal{R}$ - module by admitting the scalar multiplication $\mathcal{C}_{r \times a\mathcal{L}}(s) = \mathcal{C}_{(r \times a)\mathcal{L}}(s)$.

Proof. From Proposition, $\mathcal{M}/\mathcal{L}$ is a group and it is abelian. For $r \in \mathcal{R}, a \in \mathcal{M}, r \times a \in \mathcal{M}$ and also distributive. Since $\mathcal{M}$ is a module. If 1 is the unit element of $\mathcal{R}, \mathcal{C}_{1 \times a\mathcal{L}}(s) = \mathcal{C}_{(1 \times a)\mathcal{L}}(s) = \mathcal{C}_{a\mathcal{L}}(s)$. Hence $\mathcal{M}/\mathcal{L}$ is an $\mathcal{R}$ - module.

Definition 3.21. Let $\mathcal{M}$ be an $\mathcal{R}$-module and $\mathcal{L} \in ML[\mathcal{RM}]$. The $\mathcal{R}$ module $\mathcal{M}/\mathcal{L}$ is known as **quotient module**.

Theorem 3.22. Let $\mathcal{K}$ and $\mathcal{L}$ be two mset modules drawn from an $\mathcal{R}$ module $\mathcal{M}$ and $f : \mathcal{K} \to \mathcal{L}$ is an mset onto homomorphism. Then, $\mathcal{M}/\text{Ker } f$ is isomorphic to Im f.

Proof. Consider a function ϕ from the root set of $\mathcal{M}/\text{Ker } f$ to the root set of Imf by $\phi(a\text{Ker}f) = f(a)$, for $a \in \mathcal{M}$.

By letting, $C_{\mathcal{M}}/Ker\ f\ (aKerf) = \mathcal{C}_{Imf}\phi(aKerf)$, we have ϕ is an mset homomorphism from $\mathcal{M}/\text{Ker}f$ to Im f. The proof of this is as follows.

$$\begin{aligned} \phi(a \text{ Ker } f + b \text{ Ker } f) &= \phi((a+b) \text{ Ker } f \\ &= f(a+b) = f(a) + f(b) \\ &= \phi(a \ker f) + \phi(b \ker f) \\ \phi(r \times a \text{ Ker } f) &= \phi((r \times a) \text{ Ker } f) \\ &= f(r \times a) = r \times f(a) \\ &= r \times \phi(a \text{ Ker } f) \\ \mathcal{C}_{\text{Ker } f}(a \text{ Ker } f) &= \mathcal{C}_{\text{Ker } f}(a) \\ &= \mathcal{C}_{\mathcal{K}}(a) = \mathcal{C}_{\mathcal{L}}(f(a) \\ &= \mathcal{C}_{\mathcal{K}}(a) = \mathcal{C}_{\mathcal{L}}(f(a) \\ &= \mathcal{C}_{Imf}(f(a) \\ &= \mathcal{C}_{Imf}(\phi(a \text{ Ker } f). \end{aligned}$$

Now, to show that ϕ is one to one, let $\phi(a \text{ Ker } f) = \phi(b \text{ Ker } f)$, for some $a, b \in \mathcal{M}$. Then, $f(a) = f(b) \Rightarrow ab^{-1} \in Kerf$. Hence, a Ker $f = b$ Ker f. So, $\mathcal{M}/\text{Ker } f$ is isomorphic to Im f.

Proposition 3.23. $\mathcal{L} \in ML[\mathcal{M}]$ if and only if $\mathcal{L}$ is an mset abelian group drawn from $\mathbb{Z}$.

Proof. The proof follows straightaway from the definitions of $ML[\mathbb{Z}\mathcal{M}]$ and mset abelian group.

4. Conclusion

Multiset modules are introduced and studied in this paper. One can further consider modules over a semi-ring structure. It is well known that modules over rings are abelian groups, but modules over semi-rings are just commutative monoids. Exploring these in a multiset context needs to be carried out, which may eventually advance further generalization of the concept of vector space incorporating the semi-rings structures, which are multisets with many applications in theoretical computer science. Modules over near-rings, a non-abelian generalization in a multiset context, are another area that needs attention. Over near-rings, one can consider near-ring modules, a non-abelian generalization of modules.

Conflicts of Interest

The authors declare that they have no conflicts of interest to this work.

References

Awolola, J. A., & Ibrahim, A. M. (2016). Some results on multi-groups. *Quasigroups and Related Systems*, *24*(2), 169–177.

Blizard, W. D. (1989). Real-valued multisets and fuzzy sets. *Fuzzy Sets and Systems*, *33*(1), 77–97. https://doi.org/10.1016/0165-0114(89)90218-2

Blizard, W. D. (1991). The development of multiset theory. *Modern Logic*, *1*(4), 319–352.

Ejegwa, P. A. (2017). Upper and lower cuts of multigroups. *Prajna International Journal of Mathematical Sciences and Applications*, *1*(1), 19–26.

Girish, K. P., & John, S. J. (2009). Relations and functions in multiset context. *Information Sciences*, *179*(6), 758–768. https://doi.org/10.1016/j.ins.2008.11.002

Ibrahim, A. M., & Ejegwa, P. A. (2016). A Survey on the Concept of Multigroups. *Journal of the Nigerian Association of Mathematical Physics*, *38*, 1–8.

Knuth, D. E. (2014). *Art of computer programming, volume 2: Seminumerical algorithms*. USA: Addison-Wesley.

Li, Y., Wang, X., & Yang, L. (2013). A study of (λ, μ)-fuzzy subgroups. *Journal of Applied Mathematics*, *2013*. https://doi.org/10.1155/2013/485768

Nazmul, S., Majumdar, P., & Samanta, S. K. (2013). On multisets and multigroups. *Annals of Fuzzy Mathematics and Informatics*, *6*(3), 643–656.

Radoaca, A. (2015). *Properties of multisets compared to sets*. In *2015 17th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing*, 187–188. https://doi.org/10.1109/SYNASC.2015.37

Rajarajeswari, P., & Uma, N. (2013). On distance and similarity measures of intuitionistic fuzzy multi set. *IOSR Journal of Mathematics*, *5*(4), 19–23. https://doi.org/10.9790/5728-0541923

Rosenfeld, A. (1971). Fuzzy groups. *Journal of Mathematical Analysis and Applications*, *35*(3), 512–517. https://doi.org/10.1016/0022-247X(71)90199-5

Sebastian, S., & Ramakrishnan, T. V. (2011). Multi-fuzzy sets: An extension of fuzzy sets. *Fuzzy Information and Engineering*, *3*(1), 35–43. https://doi.org/10.1007/s12543-011-0064-y

Sebastian, S., Jacob, M. K., Mary, V. M., & Daise, D. M. (2012). On fuzzy ideals and rings. *Journal of Computer and Mathematical Sciences*, *3*(1), 115–120.

Shravan, K., & Tripathy, B. C. (2019). Multiset mixed topological space. *Soft Computing*, *23*(20), 9801–9805. https://doi.org/10.1007/s00500-019-03831-9

Singh, D., Ibrahim, A. M., Yohanna, T., & Singh, J. N. (2007). An overview of the applications of multisets. *Novi Sad Journal of Mathematics*, *37*(3), 73–92. http://eudml.org/doc/226431

Suma, P., & John, S. J. (2020). Multiset approach to algebraic structures. In C. Jana., T. Senapati & M. Pal (Eds.), *Handbook of research on emerging applications of fuzzy algebraic structures* (pp. 78–90). IGI Global Publishing. https://doi.org/10.4018/978-1-7998-0190-0

Tella, Y., & Daniel, S. (2013). A study of group theory in the context of multiset theory. *International Journal of Science and Technology*, *2*(8), 609–615.

Tripathy, B. C., Debnath, S., & Rakshit, D. (2018). On multiset group. *Proyecciones (Antofagasta)*, *37*(3), 479–489. https://doi.org/10.4067/S0716-09172018000300479

Wildberger, N. J. (2003). *A new look at multisets. School of Mathematics, UNSW Sydney*, 1–21.

Yager, R. R. (1986). On the theory of bags. *International Journal of General System*, *13*(1), 23–37. https://doi.org/10.1080/03081078608934952

Zadeh, L. A. (1965). Information and control. *Fuzzy Sets*, *8*(3), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

How to Cite: Jacob John, S., P, S., & T. M., A. (2022). Multiset Modules. *Journal of Computational and Cognitive Engineering* 1(1), 37–41, https://doi.org/10.47852/bonviewJCCE6152118205514